不协调知识的缺省推理*

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Default Reasoning with Inconsistent Knowledge

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Received 2003-12-26; Accepted 2004-03-17

Han Q, Lin ZQ. Default reasoning with inconsistent knowledge. ***Journal of Software*****, 2004,15(7):1030~1041.** http://www.jos.org.cn/1000-9825/15/1030.htm

Abstract: A novel theory called bi-default theory is proposed for handling inconsistent knowledge simultaneously in the context of default logic without leading to triviality of the extension. To this end, the positive and negative transformations of propositional formulas are defined such that the semantic link between a literal and its negation is split. Most theorems of default logic can be reproduced in the setting of the bi-default logic. It is proven that the bi-default logic is a generalization of the default logic in the presence of inconsistency. A method is provided as an alternative approach for making the reasoning ability of paraconsistent logic as powerful as the classical one.
Key words: default logic; paraconsistent logic; four-valued logic; bi-default theory

摘 要: 提出了一个新的缺省推理理论,称为双缺省理论,使得缺省逻辑在四值语义下能够同时处理不协调的知识而不导致扩张的平凡性.为此,定义了命题公式的正变换和负变换,以便分离一个文字与其否定的语义联系.大多数关于缺省逻辑的定理都可以在双缺省逻辑下重建,证明了双缺省逻辑是缺省逻辑在不协调情形下的一般化.提供了一种方法使得超协调逻辑能够获得类似经典逻辑的推理能力.
关键词: 缺省逻辑;超协调逻辑;四值逻辑;双缺省理论

中图法分类号: TP18 文献标识码: A

1 Introduction

The reasoning systems of classical logic suppose to reasoning with consistent knowledge; otherwise, a single contradiction may destroy the vast amount of meaningful knowledge. Even if the pursuit of consistency, nonmonotonic reasoning has also the problem when faced with inconsistency. Default logic[1] is a widely

* Supported by the National Natural Science Foundation of China under Grant Nos.69925203, 60373002 (国家自然科学基金)

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investigated formalism of nonmonotonic reasoning. In the context of default logic, it is well-known that once the set of axioms of a default theory is inconsistent, the default extension will collapse into triviality immediately. Theoretically, nonmonotonic logic in general and default logic in particular may lead to inconsistency[2]. On the other hand, it is advisable to introduce paraconsistency to conquer the trivial problem of reasoning in the presence of inconsistency. Some formalizations of paraconsistent and nonmonotonic reasoning have been proposed, in which a common technique is by appeal to multiple-valued logics, in particular a four-valued logic ([3~8], among others). However, it will take much effort to use a multiple-valued logic directly as the underlying logic of the default theory.

In this paper, we investigate the issue of simultaneously handling inconsistent information and consistently revising beliefs in the context of default logic. A technique called bi-default theory is developed to reason with inconsistent knowledge which allows the set of axioms of a default theory to be inconsistent. Compared with Reiter's original formalism, the bi-default theory does not lead to triviality. Technically, we transform a default theory T into a pair $T^B=(T^+,T^-)$. Though T may be inconsistent, both T^+ and T^- are always consistent. Consequently, the truth value of a formula φ comes from two parts: one is the positive part $\overline{\varphi}^+$ according to T^+; the other is the negative part $\overline{\neg\varphi}^-$ according to T^-. Indeed, we have a classical two-valued semantics for the formula of the default theory in the viewpoint of the four-valued setting. Thus, the bi-default logic is both paraconsistent and nonmonotonic. The bi-default logic can be regarded as a formalization of commonsense reasoning with inconsistent and incomplete knowledge.

An advantage of the technique behind the bi-default theory is that the underlying logic of the bi-default theory is still classical two-valued logic and thus naturally enjoys the nice properties of classical logic. Another advantage is that it improves the reasoning ability of Belnap's four-valued logic[3,4]. As well known, Belnap's four-valued logic is strictly weaker than the classical logic even in the case of consistent theories. For instance, the disjunctive syllogism: φ, $\neg\varphi\vee\phi$ implying ϕ, does not hold in the four-valued logic. To resolve this weakness, Priest[9] first proposed the solution by introducing nonmonotonicity into a paraconsistent logic. In our setting, the disjunctive syllogism works well in the consistent premise, but is effectively blocked in the case of inconsistent theories without appealing to nonmonotonicity. This method gives a novel syntactic approach for reasoning from the inconsistent theories as well.

The rest of this paper is organized as follows. In Section 2, we review Reiter's default logic. In Section 3, two transformations are presented for transforming a propositional formula φ to its counterparts $\overline{\varphi}^+$ and $\overline{\varphi}^-$. In Section 4, we introduce the bi-default theory. In Section 5, we discuss related works. Finally, we make conclusion in Section 6.

2 Default Logic

Through out this paper, let $\mathcal{L}$ be a propositional language. A theory is a set of formulas in $\mathcal{L}$. We write Th and $\vdash$ for the consequence operator and provability relation.

In Reiter's default logic, a *default* is an expression of the form

$$\frac{\alpha:\beta_1,\ldots,\beta_k}{\gamma},$$

where α, $\beta_1,\ldots$, β_k and γ are formulas in $\mathcal{L}$. α is said the *prerequisite*, $\beta_1,\ldots$, β_k the *justifications* and γ the *consequent* of a default. A *default theory* is defined as a pair $T=(W,D)$, where W is a set of formulas and D is a set of defaults. A default is said *normal* if it is of the form $\frac{\alpha:\gamma}{\gamma}$, *prerequisite-free* if it is of the form $\frac{:\beta_1,\ldots,\beta_k}{\gamma}$ and *prerequisite-free normal* if it is of the form $\frac{:\gamma}{\gamma}$. $T=(W,D)$ is said a *normal default theory* (resp. *prerequisite-free normal default theory*) if every default $d\in D$ is normal (resp. prerequisite-free normal).

A set E of formulas in $\mathcal{L}$ is an *extension* of $T=(W,D)$ if it is a fixed point of the operator Γ, i.e. $E=\Gamma(E)$, where Γ is defined as follows: Given a set of formulas S, $\Gamma(S)$ is the smallest set of formulas such that

(D1) $\Gamma(S)=Th(\Gamma(S))$

(D2) $W\subseteq\Gamma(S)$

(D3) If $(\alpha:\beta_1,...,\beta_k/\gamma)\in D$, $\alpha\in\Gamma(S)$ and $\neg\beta_1\notin S,\ldots,\neg\beta_k\notin S$, then $\gamma\in\Gamma(S)$.

A default theory may have none, one or multiple extensions in general. By $ext(W,D)$ we denote the family of all extensions of a default theory $T=(W,D)$. The set of *generating defaults for E wrt T*, written $GD(E,T)$, is defined by $GD(E,T)=\{\,(\alpha:\beta_1,...,\beta_k/\gamma)\in D \mid \alpha\in E$ and $\neg\beta_1\notin E,\ldots,\neg\beta_k\notin E\,\}$. $CONSEQUENTS(GD(E,T))$ denotes the set of consequents of the defaults from $GD(E,T)$.

Proposition 2.1.[1] A default theory $T=(W,D)$ has an inconsistent extension iff W is inconsistent.

Proposition 2.2.[1] If E is an extension of a default theory $T=(W,D)$, then

$$E=Th(W\cup CONSEQUENTS(GD(E,T))).$$

Let $T=(W,D)$ be a default theory. D^w denotes the set $\left\{\frac{:\varphi}{\varphi}\mid\varphi\in W\right\}$ i.e. the set of prerequisite-free normal default form of the axioms of the default theory T.

Marek, Treur and Truszczyński[10] described the family of extensions of an arbitrary prerequisite-free normal default theory as follows.

Proposition 2.3.[10] Let W, $\Psi\subseteq\mathcal{L}$. Let $D=\left\{\frac{:\varphi}{\varphi}\mid\varphi\in\Psi\right\}$. If W is inconsistent, then $ext(W,D)=\{\mathcal{L}\}$. Otherwise, $ext(W,D)$ is exactly the family of all theories of the form $Th(W\cup\Phi)$, where Φ is a maximal subset of Ψ such that $W\cup\Phi$ is consistent.

According to Proposition 2.3, $T=(W,D)$ and $T=(W,D^w\cup D)$ have the same extensions. Without loss of generality, we can assume that all default theories have the form $T=(W,D^w\cup D)$, and abbreviate it to $T=(W,D)$.

3 Transformations

We firstly give a brief review of the transforming technique proposed by Arieli in Ref.[11]. Let φ be a formula in $\mathcal{L}$. Define the scope of a negation operator $\neg$ in the formula $\neg\varphi$ as the set of all occurrences of propositional symbols in φ. An occurrence of atomic formula p in φ is *positive*, if it appears in the scope of an even number of negation operators in φ; otherwise, it is *negative*. Note that Arieli's transformation needs all formulas to be written in their logically equivalent negation normal form. In Ref.[12], Besnard and Schaub gave a more general definition by the notion of polarity.

Arieli's transformation is defined as follows: Let φ be a formula in $\mathcal{L}$. Substitute every positive occurrence in φ of an atomic formula p by a new symbol p^+, and every negative occurrence in φ of an atomic formula p by $\neg p^-$, then the resulting formula is denoted by $\overline{\varphi}$. The language obtained from $\mathcal{L}$ by Arieli's transformation is denoted by $\overline{\mathcal{L}}$.

We use the similar notations of Arieli's transformation and define two transformations as follows.

Definition 3.1. Let φ be a formula in $\mathcal{L}$. The *positive transformation* (*p-trans*, for short) is to substitute every positive occurrence in φ of an atomic formula p by a new symbol p^+, and every negative occurrence in φ of an atomic formula p by $\neg p^-$. The resulting formula is denoted by $\overline{\varphi}^+$. The *negative transformation* (*n-trans*, for short) is to substitute every positive occurrence in φ of an atomic formula p by a new symbol $\neg p^-$, and every negative occurrence in φ of an atomic formula p by p^+. The resulting formula is denoted by $\bar{\varphi}^-$.

The language obtained from $\mathcal{L}$ by the transformations defined in Definition 3.1 is still denoted by $\bar{\mathcal{L}}$.

Example 3.2. Let $\varphi = \neg(p \vee \neg q) \vee \neg q$, then

$$\bar{\varphi}^+ = \neg(\neg p^- \vee \neg q^+) \vee \neg\neg q^- = (p^- \wedge q^+) \vee q^-$$

and

$$\bar{\varphi}^- = \neg(p^+ \vee \neg\neg q^-) \vee \neg q^+ = (\neg p^+ \wedge \neg q^-) \vee \neg q^+.$$

Definition 3.3. Given a two-valued valuation v of the atomic formula p in $\mathcal{L}$, $\bar{v}$ denotes the corresponding valuation on the atomic formulas p^+ and p^- of $\bar{\mathcal{L}}$, such that $\bar{v}$ interprets p^+ as $v(p)$ and p^- as $\neg v(p)$.

Thus, the valuation $\bar{v}$ is a two-valued valuation of $\bar{\mathcal{L}}$.

Given a propositional theory Δ, Δ^+ represents the set $\{\bar{\varphi}^+ \mid \varphi \in \Delta\}$, and Δ^- the set $\{\bar{\varphi}^- \mid \varphi \in \Delta\}$. $\Delta^\pm$ denotes $\Delta^+ \cup \Delta^-$.

It is clear that p-trans makes Δ^+ be classically equivalent to a formula in which negation does not occur, and n-trans makes Δ^- be classically equivalent to a formula in which there is a single occurrence of negation in front of each atomic formula p^+ (or p^-). Therefore, given two valuations $\bar{v}_1$ and $\bar{v}_2$ such that $\bar{v}_1$ assigns *true* to every atomic formula occurring in Δ^+ and $\bar{v}_2$ assigns *false* to every atomic formula occurring in Δ^-, respectively, we may readily check that $\bar{v}_1$ is a classically consistent model of Δ^+ (resp. $\bar{v}_2$ is a classically consistent model of Δ^-); in other words, both Δ^+ and Δ^- are always consistent.

By induction on the structure of formulas in $\mathcal{L}$ in a straightforward way, it is trivial to prove the following propositions.

Proposition 3.4. Let φ be a formula in $\mathcal{L}$. If $\bar{\varphi}^+$ is the resulting formula of *p*-trans of φ, then $\neg\bar{\varphi}^+$ is the resulting formula of n-trans of $\neg\varphi$. If $\bar{\varphi}^-$ is the resulting formula of n-trans of φ, then $\neg\bar{\varphi}^-$ is the resulting formula of p-trans of $\neg\varphi$, i.e., $\neg\bar{\varphi}^+ = \overline{\neg\varphi}^-$ and $\neg\bar{\varphi}^- = \overline{\neg\varphi}^+$.

Proposition 3.5. Let φ be a formula in $\mathcal{L}$, then $v(\varphi) = \bar{v}(\bar{\varphi}^+) = \bar{v}(\bar{\varphi}^-)$.

Here are more properties of *p*/*n*-trans.

Theorem 3.6. Let Δ be a propositional theory. Δ is consistent iff $\Delta^\pm$ is consistent.

Proof. Immediately it follows from Proposition 3.4. □

Theorem 3.7. Let Δ be a consistent propositional theory, and φ is a formula in $\mathcal{L}$. If $\Delta^\pm \vdash \bar{\varphi}^+$ or $\Delta^\pm \vdash \bar{\varphi}^-$, then $\Delta \vdash \varphi$.

Proof. For every model v of Δ, by Proposition 3.5, $\bar{v}$ is the model of $\Delta^\pm$. By the completeness of propositional logic and $\Delta^\pm \vdash \bar{\varphi}^+$, we have $\bar{v}(\bar{\varphi}^+) = true$, and by $\Delta^\pm \vdash \bar{\varphi}^-$, we have $\bar{v}(\bar{\varphi}^-) = true$. By Proposition 3.5 again, $v(\varphi) = true$, and by the completeness of propositional logic, $\Delta \vdash \varphi$.

Theorem 3.8. Let Δ be a consistent propositional theory, and φ is a formula which is not a tautology in $\mathcal{L}$. If $\Delta \vdash \varphi$, then $\Delta^\pm \vdash \bar{\varphi}^+$ and $\Delta^\pm \vdash \bar{\varphi}^-$.

Proof. The proof proceeds by induction on the length of a derivation for φ. Because φ is not a tautology, the basis of induction is trivial by $\varphi \in \Delta$. Suppose that the claim of the theorem holds for all formulas having derivation of length $< n$, for some $n > 1$, and let $\varphi_1, \ldots, \varphi_n$ be a derivation of $\varphi = \varphi_n$. Since φ is not a tautology, φ_n is the result of applying the inference rule m*odus ponens* to φ_i and φ_j, for $1 \le i, j < n$. By Proposition 3.4, it is readily checked that $\bar{\varphi}^\pm$ can be derived by $\bar{\varphi}_i^{\pm}$ and $\bar{\varphi}_j^{\pm}$. By the induction hypothesis, $\bar{\varphi}_1^{\pm}, \ldots, \bar{\varphi}_n^{\pm}$ is a derivation of $\bar{\varphi}_n^{\pm} = \bar{\varphi}^\pm$. Hence, we have $\Delta^\pm \vdash \bar{\varphi}^+$ and $\Delta^\pm \vdash \bar{\varphi}^-$.

Regarding p^+ and p^- as two independent atomic formulas, the reasoning ability of the single transform Δ^+ (or Δ^-) from a given propositional theory Δ is very weak. For instance, let $\Delta=\{p,\neg p\vee q\}$, then $\Delta^+=\{p^+,p^-\vee q^+\}$, $\Delta^-=\{\neg p^-,\neg p^+\vee\neg q^-\}$, and hence $\Delta^\pm=\Delta^+\cup\Delta^-=\{p^+,p^-\vee q^+,\neg p^-,\neg p^+\vee\neg q^-\}$. It is clear that the disjunctive syllogism works on $\Delta^\pm$ but not on Δ^+ and Δ^- separately. The same issue will be further discussed in the next section as the application of a special family of the bi-default theories.

4 Bi-Default Theory

In this section, the so-called bi-default theory is defined by the application of the *p*/*n*-trans in a default theory, which can be well interpreted by a four-valued semantics. We will prove that the bi-default theory has nice properties in several respects.

Definition 4.1. Let d be a default of the form $\dfrac{\alpha:\beta_1,...,\beta_k}{\gamma}$, then $\dfrac{\overline{\alpha}^+:\overline{\beta}_1^+,...,\overline{\beta}_k^+}{\overline{\gamma}^+}$ is the p-trans result of d, denoted by $\overline{d}^+$, and $\dfrac{\overline{\alpha}^-:\overline{\beta}_1^-,...,\overline{\beta}_k^-}{\overline{\gamma}^-}$ is the n-trans result of d, denoted by $\overline{d}^-$, $\overline{d}^+$ and $\overline{d}^-$ are called *bi-defaults*. D^+ represents the set $\{\overline{d}^+ \mid d\in D\}$, and D^- the set $\{\overline{d}^- \mid d\in D\}$.

Definition 4.2. A *bi-default theory* w.r.t. the default theory $T=(W,D)$ is a pair $T^B=(T^+,T^-)$, where $T^+=(W^+,D^+)$ and $T^-=(W^-,D^-)$.

Definition 4.3. Let $T^B=(T^+,T^-)$ be a bi-default theory over a propositional language $\overline{\mathcal{L}}$. For any pair of sets of formulas $S^+,S^-\subseteq\overline{\mathcal{L}}$, let $\Gamma(S^+,S^-)$ be the pair of smallest sets of propositional formulas S'^+,S'^- from $\overline{\mathcal{L}}$ such that

(D1′) $S'^+=Th(S'^+)$ and $S'^-=Th(S'^-)$.

(D2′) $W^+\subseteq S'^+$ and $W^-\subseteq S'^-$.

(D3′) If $(\overline{\alpha}^+:\overline{\beta}_1^+,...,\overline{\beta}_k^+/\overline{\gamma}^+)\in D^+$, $\overline{\alpha}^+\in S'^+$ and $\neg\overline{\beta}_1^+\notin S^-,\dots,\neg\overline{\beta}_k^+\notin S^-$, then $\overline{\gamma}^+\in S'^+$ and $\overline{\gamma}^+\in S'^-$; If $(\overline{\alpha}^-:\overline{\beta}_1^-,...,\overline{\beta}_k^-/\overline{\gamma}^-)\in D^-$, $\overline{\alpha}^-\in S'^-$ and $\neg\overline{\beta}_1^-\notin S^+,\dots,\neg\overline{\beta}_k^-\notin S^+$, then $\overline{\gamma}^-\in S'^-$ and $\overline{\gamma}^-\in S'^+$.

A pair of sets of propositional formulas $E^B=(E^+,E^-)$, where E^+, $E^-\subseteq\overline{\mathcal{L}}$, is a *bi-extension* of T^B iff $(E^+,E^-)=\Gamma(E^+,E^-)$, i.e. iff (E^+,E^-) is a fixed point of the operator Γ.

By Proposition 3.4, $\neg\overline{\beta}_i^+=\overline{\neg\beta_i}^-$ and $\neg\overline{\beta}_i^-=\overline{\neg\beta_i}^+$ ($1\le i\le k$), so in Definition 4.3 (D3′), $\neg\overline{\beta}_i^+$ ($\neg\overline{\beta}_i^-$) is compared with S^- (resp. S^+) for consistency checking. As we have pointed out in Section 3, $\overline{\gamma}^+$ and $\overline{\gamma}^-$ are added to both S'^+ and S'^- in order to strengthen the reasoning ability of a single transform S'^+ (or S'^-). This also explains why we presuppose the set of defaults has the form $D^w\cup D$. By this assumption, when applying the bi-defaults, consistent formulas of W^+ and W^- will be mixed up, but the inconsistent ones will be kept splitting. To illustrate it, considering a simple default theory $T=(W,D)$ where $W=\{p\}$ and $D=\varnothing$ and thus $D^w\cup D=\{:p/p\}$, one may check that $E^B=(E^+,E^-)$ where $E^+=E^-=Th(\{p^+,\neg p^-\})$ is a bi-extension of T^B. The bi-default $\overline{d}^+=:p^+/p^+$ is an applicable bi-default since W^- doesn't include $\neg p^+$, then p^+ is added into both E^+ and E^-, the same is $\overline{d}^-$. But if $W=\{p,\neg p\}$, $\overline{d}^+$ is not an applicable bi-default again since W^- contains $\neg p^+$, and so E^+ contains only p^+,p^- and E^- contains only $\neg p^-,\neg p^+$. On the other hand, if $\{p^+,p^-\}\subseteq W^+$, then W must be inconsistent since at this time we have $\{p,\neg p\}\subseteq W$. The next example further explains how the bi-default theory works.

Example 4.4. Let $T^B=(T^+,T^-)$ be a bi-default theory w.r.t. the default theory $T=(W,D)$, where $W=\{z,\neg z,r\wedge q\}$ and $D=\left\{\dfrac{:z}{z},\dfrac{:\neg z}{\neg z},\dfrac{:r\wedge q}{r\wedge q},\dfrac{r:\neg p}{\neg p},\dfrac{q:p}{p}\right\}$. (One interpretation of this theory reads r as “republican”, q as “quaker”, p as “pacifist” and z is any inconsistent information.)

It is easy to see that $T^+=(W^+,D^+)$ and $T^-=(W^-,D^-)$, where

$W^+=\{z^+,z^-,r^+\wedge q^+\}$, $W^-=\{\neg z^-,\neg z^+,\neg r^-\wedge\neg q^-\}$,

$D^+=\{\frac{:z^+}{z^+},\frac{:z^-}{z^-},\frac{:r^+\wedge q^+}{r^+\wedge q^+},\frac{r^+:p^-}{p^-},\frac{q^+:p^+}{p^+}\}$, $D^-=\{\frac{:\neg z^-}{\neg z^-},\frac{:\neg z^+}{\neg z^+},\frac{:\neg r^-\wedge\neg q^-}{\neg r^-\wedge\neg q^-},\frac{\neg r^-:\neg p^+}{\neg p^+},\frac{\neg q^-:\neg p^-}{\neg p^-}\}$.

Since W is inconsistent, according to Reiter's default theory, T has only one extension $\mathcal{L}$. It is a trivial theory. But according to the bi-default theory, T^B has four bi-extensions which are given by $E_i^B=(E_i^+,E_i^-)$ ($i=1,2,3,4$), where

$$E_1^+=Th(W^+\cup\{\neg r^-\wedge\neg q^-,p^-,\neg p^+\}),\ E_1^-=Th(W^-\cup\{r^+\wedge q^+,p^-,\neg p^+\});$$
$$E_2^+=Th(W^+\cup\{\neg r^-\wedge\neg q^-,p^+,\neg p^-\}),\ E_2^-=Th(W^-\cup\{r^+\wedge q^+,p^+,\neg p^-\});$$
$$E_3^+=Th(W^+\cup\{\neg r^-\wedge\neg q^-,p^-,p^+\}),\ E_3^-=Th(W^-\cup\{r^+\wedge q^+,p^-,p^+\});$$
$$E_4^+=Th(W^+\cup\{\neg r^-\wedge\neg q^-,\neg p^+,\neg p^-\}),\ E_4^-=Th(W^-\cup\{r^+\wedge q^+,\neg p^+,\neg p^-\}).$$

Note that both E_i^+ and E_i^- ($i=1,2,3,4$) are consistent over language $\overline{\mathcal{L}}$. Intuitively, without the consideration of $\{z,\neg z\}$, E_1^B (resp. E_2^B) is the corresponding bi-extension of Reiter's original extension of the default theory T which includes $\neg p$ (resp. p); E_3^B and E_4^B are new bi-extensions which mean that both $\neg p$ and p hold in the same extension of T, therefore they are the corresponding bi-extensions of Reiter's inconsistent but non-trivial extensions (although they don't really exist in Reiter's default theory).

Belnap's structure $FOUR$[3,4] contains four truth values: the classical truth values t and f, the inconsistent truth value $\top$ and the incomplete truth value $\bot$. By means of the bi-default theory, any formula φ in the language $\mathcal{L}$ could be given a four-valued interpretation in the skeptical sense. It is worthy to note that an alternative four-valued interpretation of φ in the sense of credulity was presented in Ref.[13].

Definition 4.5. Given a default theory $T=(W,D)$, $T^B=(T^+,T^-)$ is the bi-default theory w.r.t. T, the mapping v associates a propositional formula φ with a truth value from $FOUR$ as follows:

$$v(\varphi)=\begin{cases} t & \text{if } \exists E^B \text{ s.t } \overline{\varphi}^+\in E^+ \\ f & \text{if } \exists E^B \text{ s.t } \overline{\neg\varphi}^-\in E^- \\ \bot & \text{otherwise.}\end{cases}$$

In particular, we write $v(\varphi)=\top$ iff $v(\varphi)=t$ and $v(\varphi)=f$.

Example 4.4 (*continued*). T^B has four bi-extensions E_i^B ($i=1,2,3,4$). It is easy to verify that

$$v(z)=\top,\ v(\neg z)=\top,\ v(r\wedge q)=t,\ v(p)=\top \text{ and } v(\neg p)=\top.$$

Here are some properties of the bi-default theory. In fact, many results of Reiter's default logic could be reproduced in the setting of the bi-default logic. For instance, the next theorem provides a recursive characterization of the bi-extensions.

Theorem 4.6. If $T^B=(T^+,T^-)$ is a bi-default theory w.r.t. the default theory $T=(W,D)$, then a pair of sets of propositional formulas $E^B=(E^+,E^-)$ is a bi-extension of T^B iff $E^+=\bigcup_{i=0}^{\infty}E_i^+$ and $E^-=\bigcup_{i=0}^{\infty}E_i^-$, where

$$E_0^+=W^+,\ E_0^-=W^-$$

and for $i\ge 0$

$$E_{i+1}^+=Th(E_i^+)\cup\{\overline{\gamma}^+\mid(\overline{\alpha}^+:\overline{\beta}_1^+,\dots,\overline{\beta}_k^+/\overline{\gamma}^+)\in D^+, \text{ where } E_i^+\vdash\overline{\alpha}^+ \text{ and } \neg\overline{\beta}_1^+\notin E^-,\dots,\neg\overline{\beta}_k^+\notin E^-\}$$
$$\cup\{\overline{\gamma}^-\mid(\overline{\alpha}^-:\overline{\beta}_1^-,\dots,\overline{\beta}_k^-/\overline{\gamma}^-)\in D^-, \text{ where } E_i^-\vdash\overline{\alpha}^- \text{ and } \neg\overline{\beta}_1^-\notin E^+,\dots,\neg\overline{\beta}_k^-\notin E^+\}$$
$$E_{i+1}^-=Th(E_i^-)\cup\{\overline{\gamma}^+\mid(\overline{\alpha}^+:\overline{\beta}_1^+,\dots,\overline{\beta}_k^+/\overline{\gamma}^+)\in D^+, \text{ where } E_i^+\vdash\overline{\alpha}^+ \text{ and } \neg\overline{\beta}_1^+\notin E^-,\dots,\neg\overline{\beta}_k^+\notin E^-\}$$

$\cup \{ \bar{\gamma}^- \mid (\bar{\alpha}^- : \bar{\beta}_1^-,...,\bar{\beta}_k^- / \bar{\gamma}^-) \in D^-$, where $E_i^- \vdash \bar{\alpha}^-$ and $\neg\bar{\beta}_1^- \notin E^+,...,\neg\bar{\beta}_k^- \notin E^+ \}$

Proof. Observe first that the following conditions hold:

(D1′) $\bigcup_{i=0}^{\infty} E_i^+ = Th(\bigcup_{i=0}^{\infty} E_i^+)$ and $\bigcup_{i=0}^{\infty} E_i^- = Th(\bigcup_{i=0}^{\infty} E_i^-)$.

(D2′) $W^+ \subseteq \bigcup_{i=0}^{\infty} E_i^+$ and $W^- \subseteq \bigcup_{i=0}^{\infty} E_i^-$.

(D3′) If $(\bar{\alpha}^+ : \bar{\beta}_1^+,...,\bar{\beta}_k^+ / \bar{\gamma}^+) \in D^+$, $\bar{\alpha}^+ \in \bigcup_{i=0}^{\infty} E_i^+$ and $\neg\bar{\beta}_1^+ \notin E^-,...,\neg\bar{\beta}_k^+ \notin E^-$, then $\bar{\gamma}^+ \in \bigcup_{i=0}^{\infty} E_i^+$ and $\bar{\gamma}^+ \in \bigcup_{i=0}^{\infty} E_i^-$; If $(\bar{\alpha}^- : \bar{\beta}_1^-,...,\bar{\beta}_k^- / \bar{\gamma}^-) \in D^-$, $\bar{\alpha}^- \in \bigcup_{i=0}^{\infty} E_i^-$ and $\neg\bar{\beta}_1^- \notin E^+,...,\neg\bar{\beta}_k^- \notin E^+$, then $\bar{\gamma}^- \in \bigcup_{i=0}^{\infty} E_i^-$ and $\bar{\gamma}^- \in \bigcup_{i=0}^{\infty} E_i^+$.

Let $\Gamma(E^+,E^-) = (E'^+,E'^-)$, by the minimality of Γ, we have

$$E'^+ \subseteq \bigcup_{i=0}^{\infty} E_i^+ \text{ and } E'^- \subseteq \bigcup_{i=0}^{\infty} E_i^- \quad (1)$$

For the proof from left to right, assume that E^B is a bi-extension of T^B, i.e. $\Gamma(E^+,E^-) = (E^+,E^-)$, and so

$$E^+ = E'^+ \text{ and } E^- = E'^- \quad (2)$$

By a straightforward induction on i, one easily shows that $E_i^+ \subseteq E^+$ and $E_i^- \subseteq E^-$, for all $i \ge 0$. Thus, $\bigcup_{i=0}^{\infty} E_i^+ \subseteq E^+$ and $\bigcup_{i=0}^{\infty} E_i^- \subseteq E^-$, and so, by (1) and (2), $E^+ = \bigcup_{i=0}^{\infty} E_i^+$ and $E^- = \bigcup_{i=0}^{\infty} E_i^-$.

For the proof from right to left, assume that

$$E^+ = \bigcup_{i=0}^{\infty} E_i^+ \text{ and } E^- = \bigcup_{i=0}^{\infty} E_i^- \quad (3)$$

By straightforward induction on i, one may readily check that for all $i \ge 0$, $E_i^+ \subseteq E'^+$ and $E_i^- \subseteq E'^-$, and hence, $\bigcup_{i=0}^{\infty} E_i^+ \subseteq E'^+$ and $\bigcup_{i=0}^{\infty} E_i^- \subseteq E'^-$. By (1), $\bigcup_{i=0}^{\infty} E_i^+ = E'^+$ and $\bigcup_{i=0}^{\infty} E_i^- = E'^-$. In view of (3), $E^+ = E'^+$ and $E^- = E'^-$, i.e. $\Gamma(E^+,E^-) = (E^+,E^-)$ and hence E^B is a bi-extension of T^B. □

Definition 4.7. Let T^B be a bi-default theory and suppose that E^B is a bi-extension of T^B. The set of *generating bi-defaults for* E^B *w.r.t.* T^B, written $GD(E^B,T^B)$, is defined by

$$GD(E^B,T^B) = \{ (\bar{\alpha}^+ : \bar{\beta}_1^+,...,\bar{\beta}_k^+ / \bar{\gamma}^+) \in D^+ \mid \bar{\alpha}^+ \in E^+ \text{ and } \neg\bar{\beta}_1^+ \notin E^-,...,\neg\bar{\beta}_k^+ \notin E^- \}$$
$$\cup \{ (\bar{\alpha}^- : \bar{\beta}_1^-,...,\bar{\beta}_k^- / \bar{\gamma}^-) \in D^- \mid \bar{\alpha}^- \in E^- \text{ and } \neg\bar{\beta}_1^- \notin E^+,...,\neg\bar{\beta}_k^- \notin E^+ \}$$

Theorem 4.8. If $E^B = (E^+,E^-)$ is a bi-extension of a bi-default theory T^B w.r.t. $T = (W,D)$, then $E^+ = Th(W^+ \cup CONSEQUENTS(GD(E^B,T^B)))$ and $E^- = Th(W^- \cup CONSEQUENTS(GD(E^B,T^B)))$.

Proof. Denote $Th(W^+ \cup CONSEQUENTS(GD(E^B,T^B)))$ and $Th(W^- \cup CONSEQUENTS(GD(E^B,T^B)))$ by RHS^+ and RHS^-, respectively. In view of Theorem 4.6

$$E^+ = \bigcup_{i=0}^{\infty} E_i^+ \text{ and } E^- = \bigcup_{i=0}^{\infty} E_i^- \quad (4)$$

where $E_0^{\pm}$, $E_1^{\pm}$,... are specified as usual. By induction on i, it is easy to show that $E_i^+ \subseteq RHS^+$ and $E_i^- \subseteq RHS^-$, for all $i \ge 0$.

To prove that $RHS^+ \subseteq E^+$ and $RHS^- \subseteq E^-$, observe first that it suffices to show that $CONSEQUENTS(GD(E^B,T^B)) \subseteq E^+$ and $CONSEQUENTS(GD(E^B,T^B)) \subseteq E^-$.

Let $\bar{\gamma}^+$, $\bar{\gamma}^- \in CONSEQUENTS(GD(E^B,T^B))$. Thus, there exists a default $(\bar{\alpha}^+ : \bar{\beta}_1^+,...,\bar{\beta}_k^+ / \bar{\gamma}^+) \in D^+$ such that $\bar{\alpha}^+ \in E^+$, $\neg\bar{\beta}_1^+ \notin E^-,\ldots,\neg\bar{\beta}_k^+ \notin E^-$ and a default $(\bar{\alpha}^- : \bar{\beta}_1^-,\ldots,\bar{\beta}_k^- / \bar{\gamma}^-) \in D^-$ such that $\bar{\alpha}^- \in E^-$, $\neg\bar{\beta}_1^- \notin E^+,\ldots,\neg\bar{\beta}_k^- \notin E^+$, respectively. So, by (4.4), $\bar{\alpha}^+ \in E_i^+$, for some $i \geq 0$ and $\bar{\alpha}^- \in E_j^-$, for some $j \geq 0$, and hence, $\bar{\gamma}^+ \in E_{i+1}{}^+ \subseteq E^+$, $\bar{\gamma}^+ \in E_{i+1}{}^- \subseteq E^-$ and $\bar{\gamma}^- \in E_{j+1}{}^+ \subseteq E^+$, $\bar{\gamma}^- \in E_{j+1}{}^- \subseteq E^-$. For the arbitrariness of $\bar{\gamma}^+$ and $\bar{\gamma}^-$, the conclusion holds. □

Corollary 4.9. Given a bi-default theory T^B w.r.t. $T=(W,D)$, if $E^B=(E^+,E^-)$ is a bi-extension of T^B, then both E^+ and E^- are consistent.

Proof. Assume to the contrary that either E^+ or E^- is inconsistent. If E^+ is inconsistent, since E^+ is a deductively closed set, $E^+=\mathcal{L}$, any $\bar{d}^- = (\bar{\alpha}^- : \bar{\beta}_1^-,\ldots,\bar{\beta}_k^- / \bar{\gamma}^-) \in D^-$ can not be applied for $\neg\bar{\beta}_i^- \in E^+$ ($i=1,\ldots,k$), and thus by Definition 4.7, $GD(E^B,T^B)=\{(\bar{\alpha}^+ : \bar{\beta}_1^+,\ldots,\bar{\beta}_k^+ / \bar{\gamma}^+) \in D^+ \mid \bar{\alpha}^+ \in E^+$ and $\neg\bar{\beta}_1^+ \notin E^-,\ldots,\neg\bar{\beta}_k^+ \notin E^-\}$. Therefore, $W^+ \cup CONSEQUENTS(GD(E^B,T^B))$ includes only the p-trans resulting formulas which are classically equivalent to a formula in which negation does not occur, thus by Theorem 4.8, $E^+=Th(W^+ \cup CONSEQUENTS(GD(E^B,T^B)))$ is consistent, a contradiction. When assuming E^- is inconsistent, the proof is similar.

Definition 4.10. Given bi-extensions $E^B=(E^+,E^-)$ and $F^B=(F^+,F^-)$,

$$E^B=F^B \text{ iff } E^+=F^+ \text{ and } E^-=F^-,$$

$$E^B \subseteq F^B \text{ iff } E^+ \subseteq F^+ \text{ and } E^- \subseteq F^-.$$

The next theorem is the maximality of the bi-extensions.

Theorem 4.11. If $E^B=(E^+,E^-)$ and $F^B=(F^+,F^-)$ are two bi-extensions of a bi-default theory T^B such that $E^B \subseteq F^B$, then $E^B=F^B$.

Proof. Let ($E_0{}^+, E_0{}^-, E_1{}^+, E_1{}^-, \ldots$) and ($F_0{}^+, F_0{}^-, F_1{}^+, F_1{}^-, \ldots$) be sequences of sets of formulas defined as those in Theorem 4.6, for E^B and F^B, respectively. Thus

$$E^+=\bigcup_{i=0}^{\infty}E_i^+ \text{ and } E^-=\bigcup_{i=0}^{\infty}E_i^- \tag{5}$$

$$F^+=\bigcup_{i=0}^{\infty}F_i^+ \text{ and } F^-=\bigcup_{i=0}^{\infty}F_i^- \tag{6}$$

By easy induction on i, one may verify that $F_i^+ \subseteq E_i^+$ and $F_i^- \subseteq E_i^-$, for all $i \geq 0$. Thus, by (5) and (6), $F^+ \subseteq E^+$ and $F^- \subseteq E^-$, and so $E^+=F^+$ and $E^-=F^-$, that is $E^B=F^B$. □

Similar to the default theory, a bi-default theory may have none, one or multiple bi-extensions. Example 4.4 is an illustration for multiple bi-extensions. T^B w.r.t. $T=\left(\varnothing,\left\{\frac{:p}{q},\frac{:p}{\neg q}\right\}\right)$ has no bi-extension. T^B w.r.t. $T=(\{p\},\varnothing)$ has only one bi-extension. But for a normal bi-default theory, the bi-extension can be proven to exist.

Definition 4.12. Let T^B be a bi-default theory w.r.t. the default theory $T=(W,D)$. If T is a normal default theory, then T^B is called a *normal bi-default theory*.

Theorem 4.13. Every normal bi-default theory has a bi-extension.

Proof. Let T^B be a normal bi-default theory w.r.t. the normal default theory $T=(W,D)$. Define the sequence $E_0{}^+$, $E_0{}^-$, $E_1{}^+$, $E_1{}^-$,… of sets of propositional formulas by

$$E_0{}^+=W^+ \quad , \quad E_0{}^-=W^-$$

and for $i \geq 0$

$$E_{i+1}{}^+=Th(E_i{}^+) \cup T_i^+ \cup T_i^- \quad , \quad E_{i+1}{}^-=Th(E_i^-) \cup T_i^+ \cup T_i^-,$$

where T_i^+ and T_i^- are two maximal sets of propositional formulas satisfying the following conditions:

(1) Both $E_i^+ \cup T_i^+ \cup T_i^-$ and $E_i^- \cup T_i^+ \cup T_i^-$ are consistent.

(2) If $\overline{\gamma}^+ \in T_i^+$, then there is a bi-default $(\overline{\alpha}^+ : \overline{\gamma}^+ / \overline{\gamma}^+) \in D^+$ such that $E_i^+ \vdash \overline{\alpha}^+$ and if $\overline{\gamma}^- \in T_i^-$, then there is a bi-default $(\overline{\alpha}^- : \overline{\gamma}^- / \overline{\gamma}^-) \in D^-$ such that $E_i^- \vdash \overline{\alpha}^-$.

By denoting $E^+ = \bigcup_{i=0}^{\infty} E_i^+$ and $E^- = \bigcup_{i=0}^{\infty} E_i^-$, we claim that $E^B = (E^+, E^-)$ is a bi-extension of T^B. In view of Theorem 4.6, it suffices to show

$$T_i^+ = \{ \overline{\gamma}^+ \mid (\overline{\alpha}^+ : \overline{\gamma}^+ / \overline{\gamma}^+) \in D^+, \text{ where } E_i^+ \vdash \overline{\alpha}^+ \text{ and } \neg\overline{\gamma}^+ \notin E^- \} \quad (7)$$

and

$$T_i^- = \{ \overline{\gamma}^- \mid (\overline{\alpha}^- : \overline{\gamma}^- / \overline{\gamma}^-) \in D^-, \text{ where } E_i^- \vdash \overline{\alpha}^- \text{ and } \neg\overline{\gamma}^- \notin E^+ \} \quad (8)$$

Denoted by RHS^+ the right hand side of Eq.(7) and RHS^- the right hand side of Eq.(8). Assume to the contrary that $T_i^+ \neq RHS^+$ or $T_i^- \neq RHS^-$. If $T_i^+ \neq RHS^+$, since clearly $T_i^+ \subseteq RHS^+$, there must be a formula $\overline{\gamma}^+ \in RHS^+ - T_i^+$. By the maximality of T_i^+, the set $E_i^- \cup T_i^+ \cup T_i^- \cup \{\overline{\gamma}^+\}$ is inconsistent, and so, because $E_i^- \cup T_i^+ \cup T_i^- \subseteq E_{i+1}{}^+ \subseteq E^-$, $E^- \cup \{\overline{\gamma}^+\}$ is also inconsistent. Therefore, since E^- is deductively closed, we have $\neg\overline{\gamma}^+ \in E^-$. This contradicts $\overline{\gamma}^+ \in RHS^+ - T_i^+$. If $T_i^- \neq RHS^-$, similarly, there exists a formula $\overline{\gamma}^- \in RHS^- - T_i^-$ such that $\neg\overline{\gamma}^- \in E^+$ which is a contradiction with $\overline{\gamma}^- \in RHS^- - T_i^-$. □

Two distinct bi-extensions of a normal bi-default theory also satisfy orthogonality.

Theorem 4.14. If a normal bi-default theory $T^B = (T^+, T^-)$ has two bi-extensions $E^B = (E^+, E^-)$ and $F^B = (F^+, F^-)$, then either $E^+ \cup F^+$ or $E^- \cup F^-$ is inconsistent.

Proof. By Theorem 4.6, $E^+ = \bigcup_{i=0}^{\infty} E_i^+$ and $E^- = \bigcup_{i=0}^{\infty} E_i^-$, $F^+ = \bigcup_{i=0}^{\infty} F_i^+$ and $F^- = \bigcup_{i=0}^{\infty} F_i^-$, where E_i^+, E_i^-, F_i^+ and F_i^- are defined as usual, for $i \geq 0$. Since $E^B \neq F^B$, $E^+ \neq F^+$ or $E^- \neq F^-$. By $E_0{}^+ = F_0{}^+ = W^+$ and $E_0{}^- = F_0{}^- = W^-$, there must be an integer $i \geq 0$ such that $E_i^+ = F_i^+$, $E_i^- = F_i^-$ but $E_{i+1}{}^+ \neq F_{i+1}{}^+$ or $E_{i+1}{}^- \neq F_{i+1}{}^-$. Assume first that $E_{i+1}{}^+ \neq F_{i+1}{}^+$, thus, for some $(\overline{\alpha}^+ : \overline{\gamma}^+ / \overline{\gamma}^+) \in D^+$, we have

(1) $\overline{\gamma}^+ \in E_{i+1}{}^+$ and $\overline{\gamma}^+ \notin F_{i+1}{}^+$ or (2) $\overline{\gamma}^+ \in F_{i+1}{}^+$ and $\overline{\gamma}^+ \notin E_{i+1}{}^+$,

or for some $(\overline{\alpha}^- : \overline{\gamma}^- / \overline{\gamma}^-) \in D^-$, we have

(3) $\overline{\gamma}^- \in E_{i+1}{}^+$ and $\overline{\gamma}^- \notin F_{i+1}{}^+$ or (4) $\overline{\gamma}^- \in F_{i+1}{}^+$ and $\overline{\gamma}^- \notin E_{i+1}{}^+$.

Assume that (1) holds. So, $E_i^+ \vdash \overline{\alpha}^+$ and hence $F_i^+ \vdash \overline{\alpha}^+$. But if $F_i^+ \vdash \overline{\alpha}^+$ and $\overline{\gamma}^+ \notin F_{i+1}{}^+$, then $\neg\overline{\gamma}^+ \in F^-$. On the other hand, $\overline{\gamma}^+ \in E_{i+1}{}^+$ implies $\overline{\gamma}^+ \in E_{i+1}{}^-$, by $E_{i+1}{}^- \subseteq E^-$, $\overline{\gamma}^+ \in E^-$. Thus $E^- \cup F^-$ is inconsistent.

Assume that (3) holds. So, $E_i^- \vdash \overline{\alpha}^-$ and hence $F_i^- \vdash \overline{\alpha}^-$. But if $F_i^- \vdash \overline{\alpha}^-$ and $\overline{\gamma}^- \notin F_{i+1}{}^+$, then $\neg\overline{\gamma}^- \in F^+$. On the other hand, by $\overline{\gamma}^- \in E_{i+1}{}^+$ and $E_{i+1}{}^+ \subseteq E^+$, $\overline{\gamma}^- \in E^+$. Thus $E^+ \cup F^+$ is inconsistent.

Similarly, in case of (2) we have $E^- \cup F^-$ is inconsistent and in case of (4) we have $E^+ \cup F^+$ is inconsistent.

If $E_{i+1}{}^- \neq F_{i+1}{}^-$, the claim of the theorem is easily verified to hold by the same deduction. □

The following theorem gives the relation between the default extension of a default theory and the bi-extension of a bi-default theory. For any default theory $T = (W, D)$, under restriction conditions, every default extension of the default theory corresponds to a bi-default extension of the corresponding bi-default theory, in other words, the bi-default logic is a generalization of Reiter's default logic in the presence of inconsistency.

Theorem 4.15. Let $T = (W, D)$ be a default theory such that W is consistent and every default $(\alpha : \beta_1, \ldots, \beta_k / \gamma)$ from D is prerequisite-free and $\neg\beta_1, \ldots, \neg\beta_k$, γ are not tautologies. If $E = Th(W \cup CONSEQUENTS(GD(E, T)))$ is an extension of T, then $E^B = (\Lambda, \Lambda)$ is a bi-extension of the bi-default theory T^B w.r.t. T, where $\Lambda = Th(W^{\pm} \cup CONSEQUENTS^{\pm}(GD(E, T)))$.

Proof. Observe first that $E^B=(\Lambda,\Lambda)$ satisfies:

(D1′) $\Lambda=Th(\Lambda)$

(D2′) $W^+\subseteq\Lambda$ and $W^-\subseteq\Lambda$.

For any default $\bar{d}^+=(\bar{\alpha}^+:\bar{\beta}_1^+,...,\bar{\beta}_k^+/\bar{\gamma}^+)\in D^+$ and $\bar{d}^-=(\bar{\alpha}^-:\bar{\beta}_1^-,...,\bar{\beta}_k^-/\bar{\gamma}^-)\in D^-$, if $\bar{\alpha}^+$, $\bar{\alpha}^-\in\Lambda$, $\neg\bar{\beta}_1^+$, $\neg\bar{\beta}_1^-\notin\Lambda,\dots,\neg\bar{\beta}_k^+$, $\neg\bar{\beta}_k^-\notin\Lambda$ then by Theorem 3.7, Theorem 3.8 and $\neg\beta_i$ ($1\le i\le k$) are not tautologies, there exists a default $d=(\alpha:\beta_1,...,\beta_k/\gamma)\in D$ satisfying $\alpha\in E$ and $\neg\beta_1\notin E,\dots,\neg\beta_k\notin E$. By $E=\Gamma(E)$, $\alpha\in\Gamma(E)$. Thus, by (D3) in section 2, we have $\gamma\in\Gamma(E)$, and hence $\gamma\in E$. Since γ is not a tautology, by Theorem 3.8, $\bar{\gamma}^+$, $\bar{\gamma}^-\in\Lambda$. So it immediately follows that Λ also satisfies

(D3′) If $(\bar{\alpha}^+:\bar{\beta}_1^+,...,\bar{\beta}_k^+/\bar{\gamma}^+)\in D^+$, $\bar{\alpha}^+\in\Lambda$ and $\neg\bar{\beta}_1^+\notin\Lambda$, …, $\neg\bar{\beta}_k^+\notin\Lambda$, then $\bar{\gamma}^+\in\Lambda$; If $(\bar{\alpha}^-:\bar{\beta}_1^-,...,\bar{\beta}_k^-/\bar{\gamma}^-)\in D^-$, $\bar{\alpha}^-\in\Lambda$ and $\neg\bar{\beta}_1^-\notin\Lambda,\dots$, $\neg\bar{\beta}_k^-\notin\Lambda$, then $\bar{\gamma}^-\in\Lambda$.

Thus, by the minimality of Γ, we have

$$\Gamma(\Lambda,\Lambda)\subseteq(\Lambda,\Lambda) \tag{9}$$

Denote $\Gamma(\Lambda,\Lambda)=(\Lambda^{\Vdash},\Lambda^{\vdash})$. Since for any $\bar{\gamma}^+$, $\bar{\gamma}^+\in CONSEQUENTS^{\pm}(GD(E,T))$, there exists $d=(\alpha:\beta_1,...,\beta_k/\gamma)\in D$ satisfying $\alpha\in E,\neg\beta_1\notin E,\dots,\neg\beta_k\notin E$, and accordingly $\bar{d}^+=(\bar{\alpha}^+:\bar{\beta}_1^+,...,\bar{\beta}_k^+/\bar{\gamma}^+)\in D^+$ and $\bar{d}^-=(\bar{\alpha}^-:\bar{\beta}_1^-,...,\bar{\beta}_k^-/\bar{\gamma}^-)\in D^-$. By Theorem 3.7, we have $\neg\bar{\beta}_1^+$, $\neg\bar{\beta}_1^-\notin\Lambda,\dots,\neg\bar{\beta}_k^+$, $\neg\bar{\beta}_k^-\notin\Lambda$. By assumption, all defaults from D is prerequisite-free, thus, by (D3′) of Definition 4.3, $\bar{\gamma}^+$, $\bar{\gamma}^-\in\Lambda^{\Vdash}$ and $\bar{\gamma}^+$, $\bar{\gamma}^-\in\Lambda^{\vdash}$, and so we have $CONSEQUENTS^{\pm}(GD(E,T))\subseteq\Lambda^{\Vdash},\Lambda^{\vdash}$. In view of $W^{\pm}\subseteq\Lambda$ is consistent (since W is consistent), by (D3′) of Definition 4.3, it is trivial that $W^{\pm}\subseteq\Lambda^{\Vdash}$ and $W^{\pm}\subseteq\Lambda^{\vdash}$. Note that $\Lambda=Th(W^{\pm}\cup CONSEQUENTS^{\pm}(GD(E,T)))$, and hence it immediately follows that

$$(\Lambda,\Lambda)\subseteq(\Lambda^{\Vdash},\Lambda^{\vdash}) \tag{10}$$

Thus, in view of (9) and (10), $(\Lambda,\Lambda)=\Gamma(\Lambda,\Lambda)$, which completes the proof of the theorem.

In fact, without the condition all defaults are prerequisite-free, Theorem 4.15 holds all the same, the reader may refer to Ref.[13] to get that detailed but tedious proof.

However, there are circumstances in which the bi-default theory T^B w.r.t. the default theory T has bi-extensions but T may not have. For example, one may check that the bi-default theory T^B w.r.t. the default theory $T=\left(\varnothing,\left\{\frac{:\neg p}{p}\right\}\right)$ has two bi-extensions $E_1^B=Th(\{p^+\})$ and $E_2^B=Th(\{\neg p^-\})$, but T has no extension. We point out that this coincides with the fact that the law of the excluded middle is not valid in Belnap's four-valued logic.

Finally, a special family of bi-default theories T^B w.r.t. $T=(W,D)$ are very attractive. In Theorem 2.3, assuming W is consistent and $\Psi=W$, we immediately get that the default theory $T=(W,D)$ has an unique extension $E=Th(W)$. In view of this, in Belnap's four-valued logic, given a theory W, we may consider its corresponding bi-default theory T^B w.r.t. $T=(W,D^w)$ and reason under T^B. By Theorem 4.15, if W is consistent, under the four-valued semantics, we shall get most of the conclusions excluding tautologies which could be derived from the classical propositional theory W. As to the inconsistent theory, the bi-default theory still gives as many conclusions as possible.

Denoted by $\vDash^4$ the four-valued consequence relation, and define $W\vDash^B\varphi$ iff $v(\varphi)\in\{t,\top\}$, where v is the mapping defined in Definition 4.5.

Proposition 4.16. $\vDash^B$ is nonmonotonic and paraconsistent.

Proof. For instance, in the following Example 4.19 $\{p,\neg p\vee q\}\vDash^B q$, but $\{p,\neg p,\neg p\vee q\}\nvDash^B q$. □

Theorem 4.17. Let W be a propositional theory. If $W \models^4 \varphi$ then $W \models^B \varphi$.

Proof. By Theorem 2.8 given by Arieli in Ref.[11], $W \models^4 \varphi$ iff $W^+ \models^2 \overline{\varphi}^+$. By Theorem 4.13, the normal bi-default theory T^B w.r.t. $T=(W,D^w)$ must have a bi-extension $E^B=(E^+,E^-)$. Since $W^+ \subseteq E^+$, we have $\overline{\varphi}^+ \subseteq Th(W^+) \subseteq Th(E^+)=E^+$, and so by Definition 4.5, $v(\varphi)\in\{t,\top\}$. Hence $W \models^B \varphi$. □

Theorem 4.18. If W is a consistent propositional theory and φ is not a tautology, then $W \models^2 \varphi$ iff $W \models^B \varphi$.

Proof. Since W is consistent, by Proposition 2.3, the default theory $T=(W,D^w)$ has a unique extension $Th(W)$. By Theorem 4.15, $E^B=(Th(W^{\pm}),Th(W^{\pm}))$ is a bi-extension of T^B w.r.t. T. Note that φ is not a tautology, then the claim of the theorem immediately follows from Theorem 3.7, Theorem 3.8 and Definition 4.5. □

By Theorems 4.17 and 4.18, it is clear that the reasoning ability of $\models^B$ is far stronger than that of the four-valued consequence relation. The following example illustrates this statement.

Example 4.19. Let $W=\{p,\neg p\vee q\}$. One may easily check that the bi-default theory T^B w.r.t. $T=(W,D^w)$ has an unique bi-extension $E^B=(E^+,E^-)$, where

$$E^+=Th(\{p^+,\neg p^-,p^-\vee q^+,\neg p^+\vee\neg q^-\}),$$
$$E^-=Th(\{p^+,\neg p^-,p^-\vee q^+,\neg p^+\vee\neg q^-\})$$

and so we have $W \models^B p$, $W \models^B \neg p\vee q$ and $W \models^B q$.

When adding $\neg p$ into W, one may readily check that

$$E^+=Th(\{p^+,p^-,p^-\vee q^+,\neg p^+\vee\neg q^-\}),$$
$$E^-=Th(\{\neg p^-,\neg p^+,p^-\vee q^+,\neg p^+\vee\neg q^-\}).$$

And so, we have $W \models^B p$, $W \models^B \neg p$, $W \models^B \neg p\vee q$ but $W \not\models^B q$.

As well-known, Belnap's four-valued logic is strictly weaker than classical logic even in the case of consistent premises. Interestingly, by the above example we have seen that the bi-default theory can be regarded as a novel technique on how to strengthen the reasoning ability of Belnap's four-valued logic. Moreover, due to the syntactic approach of the bi-default theory, it can be viewed as an alternative approach to making paraconsistent reasoning as powerful as the classical one.

5 Related Work

A similar technique like positive transformation appeared in Ref.[11], where the authors showed how multiple-valued theories can be shifted back to two-valued classical theories through a polynomial transformation. Their transformation of a formula φ is really the same as the positive part $\overline{\varphi}^+$ in our setting and based on a mapping from four-valued valuation to two-valued one. We use a new transformation for getting the negative part $\overline{\varphi}^-$ as well. In fact, the original inspiration of the bi-default theory came from the *bilattices*[8] that naturally generalizes Belnap's *FOUR*[3,4], where a pair of truth values, representing the degree of belief for or against an assertion, composes a whole judgment of the assertion.

In Ref.[12], the signed systems were introduced by transforming an inconsistent theory into a consistent one, in the same way as the positive transformation in our setting. While the semantic link between an atom and its negation was restored by appeal to default logic which at last resulted in a family of paraconsistent consequence relations. Roughly speaking, the signed systems do not aim at dealing with inconsistent default theories especially, since the defaults in signed systems are used to reestablish the context between renamed atoms and the atoms from the original theory. Nevertheless, the reader may readily verify that the signed systems have the same results as that of the bi-default theory when the latter is applied to improve the reasoning ability of the four-valued logics. In this sense, bi-default theory would be regarded as an alternative formalization of the signed systems.

In Ref.[14], a formalization of inconsistent default reasoning was proposed based on a particularly paraconsistent logic LEI. The main difference between that approach and the bi-default theory is the latter's underlying logic is still classical two-valued logic and thus enjoys the nice properties of the classical logic naturally.

6 Conclusions

Our main goal in this paper is to provide default logic with the ability for handling inconsistency and nonmonotonicity simultaneously. Thus, the bi-default theory has potential applications in the practice of commonsense reasoning in presence of inconsistency and incompleteness.

By the technique of the bi-default theory, we have successfully done. The bi-default theory can be well interpreted by a four-valued semantics. We firmly believe that most results of the default logic in the literature could be reproduced in the setting of the bi-default logic, because the bi-default logic is a generalization of Reiter's default logic under the four-valued semantics. A byproduct is that the bi-default theory can be applied to strengthen the reasoning ability of Belnap's four-valued logic, which provides an alternative approach for making multiple-valued reasoning as powerful as the classical one.

The results of this paper are limited on propositional level, we will extend it to first-order case and make a more comprehensive investigation into the bi-default theory in the future work.

References:

[1] Reiter R. A logic for default reasoning. Artificial Intelligence, 1980,13(1-2):81~132.

[2] Hanks S, McDermott D. Nonmonotonic logics and temporal projection. Artificial Intelligence, 1987,33(3):379~412.

[3] Belnap ND. A useful four-valued logic. In: Epstein G, Dunn J M, eds. Modern Uses of Multiple-Valued Logic. Dordrecht: Reidel Publishing Company, 1977. 7~37.

[4] Belnap ND. How computer should think. In: Ryle G, ed. Contemporary Aspects of Philosophy. Stocksfield: Oriel Press, 1977. 30~56.

[5] Kifer M, Lozinskii EL. A logic for reasoning with inconsistency. Journal of Automated Reasoning, 1992,9(2):179~215.

[6] Lin ZQ. Circumscription in a paraconsistent logic. Journal of Software, 1995,6(5):290~295 (in Chinese with English abstract).

[7] Arieli O, Avron A. The value of the four values. Artificial Intelligence, 1998,102(1):97~141.

[8] Ginsberg ML. Multi-Valued logics: A uniform approach to reasoning in AI. Computational Intelligence, 1988,4:256~316.

[9] Priest G. Reasoning about truth. Artificial Intelligence, 1989,39(2):231~244.

[10] Marek W, Treur J, Truszczyński M. Representation theory for default logic. Annals of Mathematics and Artificial Intelligence, 1997,21(2-4):343~358.

[11] Arieli O, Denecker M. Reducing preferential paraconsistent reasoning to classical entailment. Journal of Logic and Computation, 2003,13(4):557~580.

[12] Besnard P, Schaub T. Signed systems for paraconsistent reasoning. Journal of Automated Reasoning, 1998,20(1-2):191~213.

[13] Han Q. A study of nonmonotonic paraconsistent reasoning [Ph.D. Thesis]. Beijing: Peking University, 2004 (in Chinese with English abstract).

[14] Pequeno T, Buchsbaum A. The logic of epistemic inconsistency. In: Allen JA, Fikes R, Sandewall E, eds. Proc. of the 2nd Int'l. Conf. on the Principles of Knowledge Representation and Reasoning. San Mateo: Morgan Kaufmann Publishers, Inc., 1991. 453~560.

附中文参考文献:

[6] 林作铨.一个在弗协调逻辑中的限制.软件学报,1995,6(5):290~295.

[13] 韩庆.非单调超协调推理研究[博士学位论文].北京:北京大学,2004.