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On the Positivity and Convexity of Polynomials*

ZHENG Jin-jin[1], CHEN Xiao-qun[2], ZHANG Jian-jun[1]

[1](*National Centre for Computer Animation, Bournemouth University, Poole, Dorset BH*12 5*BB, United Kingdom*);

[2](*Department of Mathematic, University of Science and Technology of China, Hefei* 230026, *China*)

E-mail: jj_zheng@lycos.com; xqchen@ustc.edu.cn; jzhang@bournemouth.ac.uk

http://ncca.bournemouth.ac.uk/main/staff/jian

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Abstract: The convexity of curves and surfaces is an important property in the field of Computer Aided Geometric Design (CAGD). This paper tries to tackle the positive and convex problem of polynomials. Convexity can be solved by positivity. An algorithm for the positivity of polynomials is developed by extending the classic Sturm theorem. Hence, a necessary and sufficient condition for the positivity of polynomials of arbitrary degree is presented in this paper. A practical algorithm to express this condition in terms of the coefficients of the polynomials is also given.

Key words: standard sequence; greatest common divisor; positivity; convexity; Bernstein-Polynomials

Convexity of polynomials is often considered in the designing of the curved surfaces of products such as air-crafts, ships and cars. It is well known that the convexity of a polynomial over an interval is equivalent to the positivity of its second order derivative over the same interval. By considering the positivity of the second. order derivative of polynomials, Ref.[9] presented a sufficient condition for the convexity of Bernstein polynomials over triangles. Further, Ref.[10] provided an improved convex condition. Suppose the degree of a Bernstein polynomial is n, the improved convex condition is sufficient and necessary when $n \le 3$. However, it is still an open question when $n \ge 4$, where only the sufficient condition is available to verify the convexity.

Given a polynomial $f(x), x \in (\alpha, \beta)$. Its positivity and convexity are defined as following.

Definition 1.1 (**Positivity**). *f*(*x*) is positive over an interval (α, β) if for any $x \in (\alpha, \beta)$, $f(x) \ge 0$.

Definition 1.2 (**Strict positivity**). *f*(*x*) is positive over an interval (α, β) if for any $x \in (\alpha, \beta)$, $f(x) > 0$.

Definition 1.3 (**Convexity**). *f*(*x*) is convex over an interval (α, β) if for any $x_1, x_2 \in (\alpha, \beta)$, the following formula holds

$$f\left(\frac{x_1 + x_2}{2}\right) \le \frac{f(x_1) + f(x_2)}{2}. \tag{1}$$

For a polynomial *f*(*x*), formula (1) is equivalent to the positivity of its second derivative

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ZHENG Jin-jin was born in 1963. He is a research fellow at the National Centre for Computer Animation, Bournemouth University, UK. He received his Ph.D. degree in computer aided geometric modelling from the University of Birmingham, UK in 1998. His research interests include CAGD and computer animation. **CHEN Xiao-qun** was born in 1962. She received her Ph.D. degree in computational geometry from the University of Science and Technology of China in 1999. Her research interests include computational geometry and computer graphics. **ZHANG Jian-jun** was born in 1959. He is a professor in the Computer Graphics and Director of Research Projects at the National Centre for Computer Animation, Bournemouth University, UK. He received his Ph.D. degree from the Chongqing University in 1987. His research interests include computer graphics, CAD and computer animation.

$$\frac{d^2 f(x)}{dx^2} \ge 0, \quad \text{for all} \quad x \in (\alpha, \beta)\,. \tag{2}$$

So the positivity of a polynomial is considered in this paper instead of the convexity of a polynomials. It is obvious that a polynomial $f(x)$ is positive over an interval $[\alpha, \beta]$ if and only if

(1) $f(x)$ has no roots or has the only roots with even multiplicities within the interval; and

(2) $f(x_0)>0$ for some $x_0 \in [\alpha, \beta]$.

The technique of Sturm Theorem is a conventional way to verify whether a polynomial has no roots over an interval. However, Sturm theorem cannot verify whether a polynomial has roots with even multiplicities over an interval. In this paper, a necessary and sufficient condition is given to verify whether a polynomial has roots with even multiplicities over an interval. A recursive algorithm to verify the positivity of a polynomial is also provided.

The paper is organized as follows. In Section 1, the technique of standard sequence is further exploited. Conventionally, for a polynomial $f(x)$, a standard sequence is considered to verify whether it has roots over an interval. Here, the last term of a standard sequence, i.e., the greatest common divisor of the two polynomials $f(x)$ and $f'(x)$, is considered as the starting polynomial for another standard sequence. In this way, an extended standard sequence is presented. Consequently, a necessary and sufficient condition (NASC) for the positivity of a polynomial over an interval is obtained. In Section 2, a practical algorithm is given to express the coefficients of the standard sequences in terms of the coefficients of a polynomial. In Section 3, some examples are given to demonstrate the algorithm. Section 4 concludes the paper.

1 A Necessary and Sufficient Condition for the Positivity of Polynomials

Given a degree n polynomial with real coefficients

$$f(x) = \sum_{i=0}^{n} a_i x^i\,, \quad x \in (-\infty, \infty)\,. \tag{3}$$

Using the modified Euclidean algorithm, we define the standard sequence for $f(x)$:

$$f_0(x) := f(x)\,, \quad f_1(x) := f'(x)\,, \tag{4}$$

and forming the remaining polynomials $f_{i+1}(x)$ recursively, dividing $f_{i-1}(x)$ by $f_i(x)$,

$$f_{i-1}(x) = q_i(x) f_i(x) - c_{i+1} f_{i+1}(x)\,, \quad i = 1,2,\dots,m-1\,, \quad f_{m-1}(x) = q_m(x) f_m(x)\,, \quad f_{m+1}(x) \equiv 0\,, \tag{5}$$

where $\deg f_i(x) > \deg f_{i+1}$, and the constant $c_{i+1} > 0$ are positive but otherwise arbitrary. The final polynomial $f_m(x)$ is the greatest common divisor (GCD) of $f(x)$ and $f'(x)$.

Lemma 1.1 (**Sturm's theorem**). Let $f(x)$ be a polynomial of positive degree n with coefficients in a real closed field $\boldsymbol{R}$ and let $f_0(x) = f(x)$, $f_1(x) = f'(x), \dots, f_m(x)$ be the standard sequence (1.2) for $f(x)$. Assume $[\alpha, \beta]$ is an interval such that $f(\alpha), f(\beta) \ne 0$. Then the number of distinct roots of $f(x)$ in (α, β) is $v_\alpha(f) - v_\beta(f)$, where $v_\alpha(f)$ denotes the number of variations in sign of sequence $f_0(\alpha), f_1(\alpha), \dots, f_m(\alpha)$.

A proof of Lemma 1.1 is found in Ref.[1].

So the polynomial $f(x)$ has no roots over the interval (α, β) if and only if $v_\alpha(f) - v_\beta(f) = 0$. In this case, $f(x) > 0$ for all $x \in (\alpha, \beta)$. It is said $f(x)$ is strictly positive over (α, β). However, if $f(x)$ has roots within (α, β), Sturm theorem fails to verify whether $f(x) \ge 0$ over this interval.

This drawback is overcome by further exploiting the standard sequence as following.

If $\deg f_m(x) \ge 1$, let $f^1(x) := f_m(x)$. We can also get the standard sequence $f_0^1(x)$, $f_1^1(x)$, …, $f_{m_1}^1(x)$ for $f_m(x) = f^1(x)$, namely,

$$f_0^1(x) := f^1(x), f_1^1(x) := (f^1(x))'\,,$$

$$\vdots$$

$$f_{i-1}^1(x) = q_i^1(x) f_i^1(x) - c_{i+1}^1 f_{i+1}^1(x), \quad i = 1,2,\ldots,m_1 - 1, \tag{6}$$

$$f_{m_1-1}^1(x) = q_{m_1}^1(x) f_{m_1}^1(x),\ f_{m_1+1}^1(x) \equiv 0,$$

where $\deg f_i^1(x) > \deg f_{i+1}^1(x)$, and the constants $c_{i+1}^1 > 0$ are positive but otherwise arbitrary.

Generally, if $\deg f_{m_{m-1}}^{k-1}(x) \geq 1$, let

$$f^k(x) := f_{m_{k-1}}^1(x),$$

then we can get the standard sequence $f_0^k(x), f_1^k(x), \ldots, f_{m_k}^k(x)$ for $f^k(x) = f_{m_{k-1}}^{k-1}(x)$.

It is obvious that $f^i(x)$ is the GCD of $f^{i-1}(x)$ and $(f^{i-1}(x))'$, and

$$f^{i+j}(x) = (f^i)^j(x). \tag{7}$$

We denote $f^0(x) := f(x)$. Then the main theorem is presented here.

Theorem 1.2. A NASC for the positivity of polynomials). Assume $f(\alpha)$, $f(\beta) \neq 0$, $f_{m_K}^K(x)$=constant, then $f(x) \geq 0$, $x \in (\alpha,\beta)$ or $f(x) \leq 0$, $x \in (\alpha,\beta)$, if and only if

$$v_\alpha(f^{2i}) - v_\beta(f^{2i}) = v_\alpha(f^{2i+1}) - v_\beta(f^{2i+1}), \quad i = 0,1,\ldots,\left[\frac{K-1}{2}\right]. \tag{8}$$

where $\left[\frac{K-1}{2}\right]$ denotes the maximum integer which is less than or equal to $\frac{K-1}{2}$. □

To prove Theorem 1.2, we introduce some lemmas.

Lemma 1.3. Assume x_0 is a root of $f(x)$ with multiplicity n_0. Then x_0 is a root of $f^i(x)$ with multiplicity $n_0 - i, (i < n_0)$, or is not a root of $f(x), (n_0 \leq i)$, where $f^i(x) = f_{m_{i-1}}^{i-1}$ is the final polynomial of the standard sequence for f^{i-1}.

Proof. Suppose $f(x) = (x - x_0)^{n_0} Q_0(x), Q_0(x_0) \neq 0$ Since

$f^1(x) = f_m(x)$ is the GCD of $f(x)$ and $f'(x)$,

$f^2(x) = f_{m_1}^1(x)$ is the GCD of $f^1(x)$ and $f'(x)$,

$\vdots$

$f^i(x) = f_{m_{i-1}}^{i-1}(x)$ is the GCD of $f^{i-1}(x)$ and $(f^{i-1}(x))'$.

If $i < n_0$, we get

$$f^1(x) = (x - x_0)^{n_0 - 1} Q_1(x), Q_1(x_0) \neq 0,$$

$$f^2(x) = (x - x_0)^{n_0 - 2} Q_2(x), Q_2(x_0) \neq 0,$$

$$\vdots$$

$$f^i(x) = (x - x_0)^{n_0 - i} Q_i(x), Q_i(x_0) \neq 0.$$

Hence, x_0 is a root of $f^i(x)$ with multiplicity $n_0 - i$.

If $n_0 \leq i$, then $f^i(x)$ has no root x_0.

Lemma 1.4. Assume $[\alpha,\beta]$ is an interval such that $f(\alpha), f(\beta) \neq 0$. Then $f(x)$ does not change sign in $[\alpha,\beta]$ if and only if the multiplicity of any root of $f(x)$ in $[\alpha,\beta]$ is even or $f(x)$ has no root in $[\alpha,\beta]$.

Lemma 1.5. Assume $[\alpha,\beta]$ is an interval such that $f(\alpha), f(\beta)\neq 0$. Then

$$v_\alpha(f)-v_\beta(f)=v_\alpha(f^1)-v_\beta(f^1), \tag{9}$$

if the multiplicity of any root of $f(x)$ in $[\alpha,\beta]$ is even, where $f^1(x)=f_m(x)$ is the final polynomial of the standard sequence for $f(x)$.

Proof. Let $x_j, j=1,2,...,l$ are all of the distinct roots of polynomial $f(x)$ in $[\alpha,\beta]$, n_j are the corresponding multiplicities. Since n_j are all even, we get $n_j\geq 2$.

Let $f(x)=\prod_{j=1}^{l}(x-x_j)^{n_j}Q_0(x), Q_0(x_j)\neq 0$. Since $f^1(x)=f_m(x)$ is a GCD of $f(x)$ and $f'(x)$, we have $f^1(x)=\prod_{j=1}^{l}(x-x_j)^{n_j-1}Q_1(x), Q_1(x_j)\neq 0$, $n_j-1\geq 1$.

It follows that $x_j, j=1,2,...,l$ are also roots of $f^1(x)$. It implies that the number of distinct roots of $f^1(x)=f_m(x)$ in $[\alpha,\beta]$ is equal to that of $f(x)$. From Lemma 1.1 we know that (9) holds.

Proof of Theorem 1.2.

First we prove the necessity. Suppose $f(x)\geq 0, x\in[\alpha,\beta]$, or $f(x)\leq 0, x\in[\alpha,\beta]$.

(a) If $f(x)$ has no root in $[\alpha,\beta]$, then $f^i(x)$ have no root in $[\alpha,\beta]$. From Lemma 1.1, we know that (6) holds.

(b) If $f(x)$ has roots in $[\alpha,\beta]$, by Lemma 1.4 we know, the multiplicity of any root of $f(x)$ in $[\alpha,\beta]$ is even. From Lemma 1.3 we know either $f^{2i}(x)$ have no root in $[\alpha,\beta]$, or the multiplicities of any root of $f^{2i}(x)$ in $[\alpha,\beta]$ are even, $i=0,1,...,\left[\frac{K}{2}\right]$. By Lemma 1.5 we obtain that (8) holds.

Secondly, we prove the sufficiency. Suppose (8) holds.

(a) If $f(x)$ has no root in $[\alpha,\beta]$, then from $f(\alpha), f(\beta)\neq 0$ we get $f(x)>0$, $x\in[\alpha,\beta]$, or $f(x)<0, x\in[\alpha,\beta]$.

(b) If $f(x)$ has root in $[\alpha,\beta]$, we can prove that the multiplicity of any root of $f(x)$ in $[\alpha,\beta]$ must be even. Otherwise, assume $x_0\in[\alpha,\beta]$ is a root of $f(x)$ with multiplicity $2n_0+1$, $n_0\geq 0$ is an integer.

From Lemma 1.3 we know that x_0 is a simple root of f^{2n_0}, and is not a root of f^{2n_0+1}. It follows that

$$v_\beta(f^{2n_0})-v_\beta(f^{2n_0})\geq 1+v_\beta(f^{2n_0+1})-v_\beta(f^{2n_0+1}),$$

which contradicts to (8). The contradiction implies that the multiplicity of any root of $f(x)$ in $[\alpha,\beta]$ is even. By Lemma 1.4 we know that $f(x)$ does not change sign in $[\alpha,\beta]$.

Corollary 1.6. Assume $f(\alpha), f(\beta)\neq 0$. Then $f(x)$ does not change sign for all $x\in[\alpha,\beta]$ if and only if for some $K_1, 2K_1\leq K$, $f^{2K_1}(x)$ does not change sign in $[\alpha,\beta]$, and

$$v_\alpha(f^{2i})-v_\beta(f^{2i})=v_\alpha(f^{2i+1})-v_\beta(f^{2i+1}), \quad i=0,1,...,K_1-1. \tag{10}$$

Proof. First we prove the necessity. Suppose $f(x)\geq 0, x\in[\alpha,\beta]$ or $f(x)\leq 0, x\in[\alpha,\beta]$. From Theorem 1.2 we know that (8) holds. Since (10) is part of (8), we get (10) holds. By Lemma 2.4 we know that the multiplicity of any root of $f(x)$ is even. From Lemma 1.3 we obtain either $f^{2K_1}(x)$ has no root in $[\alpha,\beta]$, or the multiplicity of any root of $f^{2K_1}(x)$ in $[\alpha,\beta]$ is even. Using Lemma 1.4 again, we obtain that $f^{2K_1}(x)$ does not change sign in $[\alpha,\beta]$.

Secondly, we prove the sufficiency. Suppose $f^{2K_1}(x)$ does not change sign and (10) holds.

Since f^{2K_1} does not change sign, by the same process of the proof of Theorem 1.2 the following holds

$$v_\alpha(f^{2i})-v_\beta(f^{2i})=v_\alpha(f^{2i+1})-v_\beta(f^{2i+1}),\quad i=K_1,K_1+1,...,\left[\frac{K-1}{2}\right]. \tag{11}$$

From (10) and (11) we know that (8) holds. It follows that $f(x)\ge 0, x\in[\alpha,\beta]$, or $f(x)\le 0$, $x\in[\alpha,\beta]$.

From Lemma 1.1 it is obvious that

Proposition 1.7. Suppose $f(\alpha), f(\beta)>0(<0)$. Then $f(x)>0(<0),\ x\in[\alpha,\beta]$ if and only if $v_\alpha(f)-v_\beta(f)=0$.

Remarks: Theorem 1.2, Corollary 1.6 and Proposition 1.7 can practically be used to justify whether a polynomial is non-negative at any interval. In the next four sections of this paper, the author gives some applications.

2 A Practical Algorithm to Express the NASCs for the Positivity of Polynomials

Theorem 1.2 has given the necessary and sufficient conditions for the positivity of polynomials. Using it, we can justify the positivity of polynomials of any degree in arbitrary intervals.

However, for polynomials of higher degree, even using a computer program, the conditions in Theorem 1.2 are difficult to verify, since those conditions are not expressed in terms of the coefficients of the polynomials.

From Theorem 1.2, we know if the coefficients of the standard sequences (4) to (6) can be expressed in terms of the coefficients of polynomial (3), then the NASC for the positivity of polynomial (3) can be expressed in terms of the coefficients of polynomial (3).

The following is an algorithm to express the coefficients of the standard sequences (4), (5) and (6) in terms of the coefficients of polynomials (3).

Suppose

$$f(x)=\sum_{j=0}^{n}a_j x^j,\quad a_n\ne 0, \tag{12}$$

then

$$f_1(x)=\sum_{j=0}^{n-1}a_{1,j}x^j, \tag{13}$$

where $a_{1,j}=(j+1)a_j,\ j=0,1,...,n-1$.

We calculate the coefficients of $f_i(x), i=2,...,m$, recursively.

Let degree $f_{i+1}(x)=L$, degree $f_i(x)=L+P,\quad P\ge 1$, then

$$f_i(x)=a_{i,L+P}x^{L+P}+a_{i,L+P-1}x^{L+P-1}+...+a_{i,0}, \tag{14}$$

$$f_{i+1}(x)=a_{i+1,L}x^L+a_{i+1,L-1}x^{L-1}+...+a_{i+1,0}, \tag{15}$$

where $a_{i,L+P}$ and $a_{i+1,L}\ne 0$.

We want to get the coefficient of polynomial $f_{i+2}(x)$.

We denote matrices

$$A_{i+1}:=\begin{pmatrix} a_{i+1,L} & a_{i+1,L-1} & \cdots & a_{i+1,L-P} \\ & a_{i+1,L} & \cdots & a_{i+1,L-P+1} \\ & & \ddots & \vdots \\ & & & a_{i+1,L} \end{pmatrix}_{(P+1)\times(P+1)}, \tag{16}$$

$$B_{i+1} := \begin{pmatrix} a_{i+1,L-P-1} & \cdots & a_{i+1,0} & & & \\ a_{i+1,L-P} & \cdots & a_{i+1,1} & a_{i+1,0} & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ a_{i+1,L-1} & \cdots & a_{i+1,P} & a_{i+1,P-1} & \cdots & a_{i+1,0} \end{pmatrix}_{(P+1)\times L}, \tag{17}$$

where we denote $a_{i+1,j} := 0$, if $j < 0$,

$$C_i := (a_{i,L+P}, a_{i,L+P-1}, \ldots, a_{i,L})_{(P+1)\times 1}, \tag{18}$$

$$D_i := (a_{i,L-1}, a_{i,L-2}, \ldots, a_{i,0})_{L\times 1}. \tag{19}$$

Then we have

Proposition 2.1. (**An algorithm to calculate the standard sequence**). If $f_i(x)$ and $f_{i+1}(x)$ are polynomials in (14) and (15), then the polynomial

$$f_{i+2}(x) = a_{i+2,L-1}x^{L-1} + a_{i+2,L-2}x^{L-2} + \ldots + a_{i+2,0},$$

where the coefficients of $f_{i+2}(x)$ are $(a_{i+2,L-1}, a_{i+2,L-2}, \ldots, a_{i+2,0}) = C_i A_{i+1}^{-1} B_{i+1} - D_i$ and $a_{i+2,j}, j = 0,1,\ldots, L-1$ may be zeros.

Proof. From the definition (16)–(19) of matrices A_{i+1}, B_{i+1}, C_i, and D_i, we can obtain that

$$\begin{pmatrix} A_{i+1} & B_{i+1} \\ C_i & D_i \end{pmatrix} \begin{pmatrix} x^{L+P} \\ \vdots \\ x^{L-1} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} x^P f_{i+1}(x) \\ \vdots \\ x f_{i+1}(x) \\ f_{i+1}(x) \\ f_i(x) \end{pmatrix}. \tag{20}$$

Since $a_{i+1,L} \neq 0$, matrix A_{i+1} has converse A_{i+1}^{-1}.

Matrix $\begin{pmatrix} I_{P+1} & 0_{P+1,1} \\ -C_i A_{i+1}^{-1} & 1 \end{pmatrix}_{(P+2)\times(P+2)}$ multiplies (21) in the left, we obtain that

$$\begin{pmatrix} A_{i+1} & B_{i+1} \\ 0_{L,P+1} & D_i - C_i A_{i+1}^{-1} B_{i+1} \end{pmatrix} \begin{pmatrix} x^{L+P} \\ \vdots \\ x^{L-1} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} x^P f_{i+1}(x) \\ \vdots \\ x f_{i+1}(x) \\ f_{i+1}(x) \\ f_i(x) - f_{i+1}(x) C_i A_{i+1}^{-1} \begin{pmatrix} x^P \\ \vdots \\ 1 \end{pmatrix} \end{pmatrix}. \tag{21}$$

Hence,

$$(D_i - C_i A_{i+1}^{-1} B_{i+1}) \begin{pmatrix} x^{L-1} \\ \vdots \\ 1 \end{pmatrix} = f_i(x) - f_{i+1}(x) C_i A_{i+1}^{-1} \begin{pmatrix} x^P \\ \vdots \\ 1 \end{pmatrix}. \tag{22}$$

We denote polynomial $q_{i+1}(x) := C_i A_{i+1}^{-1} \begin{pmatrix} x^P \\ \vdots \\ 1 \end{pmatrix}$, then

$$(D_i - C_i A_{i+1}^{-1} B_{i+1}) \begin{pmatrix} x^{L-1} \\ \vdots \\ 1 \end{pmatrix} = f_i(x) - q_{i+1}(x) f_{i+1}(x). \tag{23}$$

By definition (5), we know that

$$f_{i+2}(x)=(C_iA_{i+1}^{-1}B_{i+1}-D_i)\begin{pmatrix}x^{L-1}\\ \vdots \\ 1\end{pmatrix}.$$

3 Examples

In this section, two examples are provided to demonstrate the algorithm presented in this paper.

Example 1.

$$f(x)=\sum_{i=0}^{n}a_ix^i:=0.289+2.305x+5.113x^2-2.458x^3-18.854x^4-8.32x^5+16.68x^6+8.4x^7+x^8.$$

By the algorithm in Proposition 2.1, we obtain the scaled coefficients of the standard sequence for the polynomial $f(x)$ as following.

	a_0	b_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
f_0^0	.289	2.305	5.11	–2.458	–18.854	–8.32	16.68	8.4	1
f_1^0	2.305	10.226	–7.373	–75.418	–41.6	100.08	58.8	8	
f_2^0	0.0135	–0.675	–4.802	–8.363	3.967	16.256	3.548		
f_3^0	–2.221	–14.408	–24.123	12.392	47.503	10.327			
f_0^1	–2.221	–14.408	–24.123	12.392	47.503	10.327			
f_1^1	–14.408	–48.247	37.176	190.013	51.634				
f_2^1	–0.43	2.649	21.315	30.001					
f_3^1	12.209	61.043	76.304						
f_0^2	12.209	61.043	76.304						
f_1^2	61.043	152.607							
f_2^2	0								

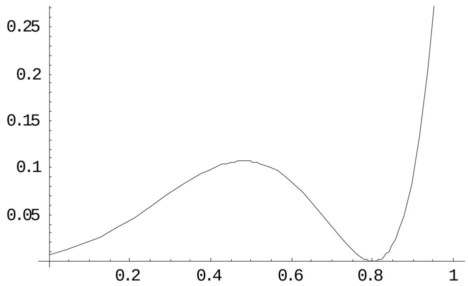

Fig.1 A nonnegative polynomial $f(x)$ in [0,1]

It is easy to verify that $V_0(f^0)-V_1(f^0)=V_0(f^1)-V_1(f^1)=1$. By Theorem 2.2 we know that $f(x)$ is nonnegative in [0,1] (see Fig.1).

Example 2.

$$f(x)=\sum_{i=0}^{n}a_ix^i:=0.297-2.566x+5.522x^2-0.397x^3-5.53x^4+5.879x^5+5.574x^6+0.93x^7+x^8.$$

The following is the scaled coefficients of the standard sequence for the polynomial $f(x)$.

	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
f_0^0	.279	–2.566	5.522	0.397	–5.53	5.879	–5.574	0.93	1
f_1^0	–2.566	11.044	1.191	–22.126	29.393	–33.444	6.51	8	
f_2^0	–0.334	2.406	–4.124	–0.570	3.193	–2.690	1.488		
f_3^0	–2.139	21.073	–46.383	–8.073	12.547	12.689			
f_4^0	1.035	–6.568	21.808	–2.222	–8.255				
f_5^0	0.993	–12.08	36.969	–22.993					
f_6^0	–0.366	1.784	–1.233						
f_7^0	0.102	–0.085							
f_0^1	0.102	–0.085							
f_1^1	–0.085								

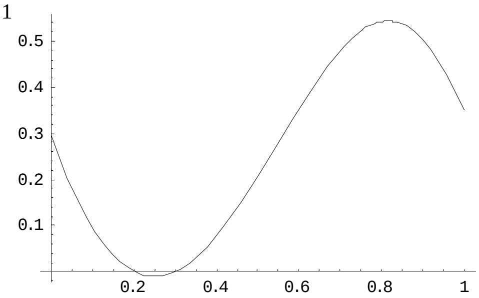

Fig.2 $f(x)$ polynomial changes sign in [0,1]

It is easy to verify that $V_0(f^0)-V_1(f^0)=2\neq V_0(f^1)-V_1(f^1)=0$. Hence by Theorem 1.2, we know that $B_{10}(x)$ changes sign in [0,1] (see Fig.2).

4 Conclusions

Convexity is an important property of polynomials, and it is often required in CAGD. It is well known that the convexity of a polynomial is equivalent to the positivity of its second derivative. By using Sturm theorem, one can verify whether a polynomial is strictly positive over an interval, i.e., whether a polynomial has roots over interval. However, Sturm theorem fails to verify whether a polynomial is positive over an interval, i.e., whether it has roots with even multiplicities.

By extending the concept of standard sequence, in this paper, a necessary and sufficient condition is presented to check whether a polynomial has roots with even multiplicities. A practical algorithm to express the condition in terms of the coefficients of the polynomial is also given.

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多项式的正性和凸性

郑津津[1], 陈效群[2], 张建军[1]

[1](Bournemouth University 国家计算机动画中心,英国);

[2](中国科学技术大学 数学系,安徽 合肥 230026)

摘要: 在计算机辅助几何设计(CAGD)中,曲面曲线的凸性是一种重要的特性.旨在解决多项式的正性和凸性问题.凸性可以通过正性来解决.通过推广经典的 Sturm 定理,得到一种多项式正性的算法.由此提出了任意阶多项式为正的一个充要条件,也提出了一个实用的算法,从而可以只用此多项式的系数来表示得到的充要条件.

关键词: 标准序列;最大公除数;正性;凸性;Bernstein-多项式

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