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多连通多边形的内部 Voronoi 图的顶点和边数的上界*

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Upper Bounds on the Size of Inner Voronoi Diagrams of Multiply Connected Polygons

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Abstract: The Voronoi diagram (VD) of a planar polygon has many applications, from path planning in robotics to collision detection in virtual reality. To study the complexity of algorithms based on Voronoi diagram, it is important to estimate the numbers of vertices and edges of a VD. Held proves that the inner Voronoi diagram of a simple polygon has at most $n+k-2$ vertices and $2(n+k)-3$ edges, where n is the number of the polygon's vertices and k is the number of reflex vertices. But this conclusion holds not for a multiply-connected polygon, i.e. a polygon with "holes". In this paper, by constructing a rooted tree from a VD, and based on some properties of the rooted tree, new upper bounds on the numbers of vertices and edges in an inner Voronoi diagram of a multiply-connected polygon are proved. The average numbers of Voronoi vertices and edges on the boundary of a VD are also presented. The result of this paper has been used to analyze the complexity of VD-based visibility computing algorithm in SDU Virtual Museum.

Key words: computational geometry; Voronoi diagram; complexity analysis; polygon; multiply connected polygon

摘 要: 多边形的 Voronoi 图在路径规划、碰撞检测等方面有着广泛的应用,其顶点和边数在这些应用算法的复杂度分析方面起着重要作用.Held 证明了一个简单多边形的内部 Voronoi 图最多有 $n+k-2$ 个顶点和 $2(n+k)-3$ 条边,其中 n 和 k 分别是多边形的顶点和内尖点数.但其结论不能适用于多连通多边形.对多连通多边形进行研究,通过将其 Voronoi 图转化为有根树,并利用有根树的性质,给出了其内部 Voronoi 图的顶点和边数上界的估计,并对 Voronoi 区域的边界所包含顶点和边数的平均值进行了讨论."SDU 数字博物馆"系统所采用的基于 Voronoi 图的可见性算法的复杂度分析,就利用了所得出的结论.

关键词: 计算几何;Voronoi 图;复杂度分析;多边形;多连通多边形
中图法分类号: TP309 文献标识码: A

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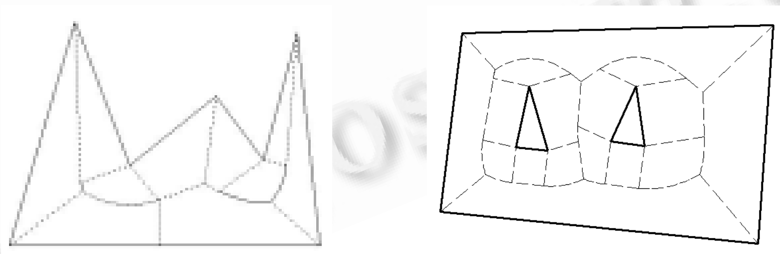
1 Introduction and Motivation

The Voronoi diagram (VD) is an important geometric structure in computational geometry. A VD records the regions in the proximity of a set of points. There is a rich literature on the Voronoi diagram of a set of points, as well as its extensions, such as high-degree Voronoi diagrams. Rather than for a set of points, the regions in the proximity of a set of objects, such as line segments, circular arcs and polygons, also form a Voronoi diagram[1−7]. The VD of a planar polygon has wide applications in pocket machining[1,8,9], path planning[10], medial axis computation[11], collision detection[12] and so on[13]. The number of vertices and edges of the VD of a polygon is important in analyzing the complexity of VD-based algorithms.

Much work has been done on analyzing the complexity of d-dimensional Voronoi diagrams of points[14−17]. It is shown[16] that the VD on n points in 2D plane has at most $2n-5$ vertices and $3n-6$ edges, and the average number of edges on the boundary of the VD does not exceed 6. For a Voronoi diagram $VD(G)$ of a planar straight line graph G on n points, Lee and Drysdale show that the number of vertices of $VD(G)$ is at most $4n-3$[18]. Reference [6] proves that the number of vertices of $VD(G)$ is exactly $2n+l+k-2$, where l and k are the number of terminals (i.e. endpoints belonging to exactly one line segment in G) and the number of reflex vertices. It is also mentioned in Ref.[6] that the inner Voronoi diagram of a simply connected polygon with n edges and k reflex vertices realizes *exactly* $n+k-2$ vertices and at most $2(n+k)-3$ edges. In fact, according to the conclusion first given by Lee in 1982[19], the Voronoi diagram of a simply connected polygon has *at most* $n+k-2$ Voronoi vertices and at most $2(n+k)-3$ edges. M. Held[1] claims that these upper bounds also hold for a multiply-connected polygon. However, we shall see that this conclusion holds only for the inner VD of a simple polygon, but not for a multiply-connected polygon, i.e. a polygon with “holes”, as shown by the following example. Figure 1 shows two kinds of polygons in solid lines and their inner Voronoi diagram in dotted lines. Let m and e denote the number of vertices and the number of edges of its inner Voronoi diagram. In Fig.1(a), n=8, k=2, m=8 and e=17, where the polygon has a flat vertex (see Definition 1). The inequalities $m\le n+k-2$ and $e\le 2(n+k)-3$ given by Held are correct. For the multiply connected polygon in Fig.1(b), n=10, k=6, m=18, and e=35. We have $m>n+k-2$ and $e>2(n+k)-3$. Thus, Held’s inequalities do not hold in this case.



(a) Simply connected polygon (b) Multiply connected polygon

Fig.1 Polygons and their inner Voronoi diagrams

We shall prove new upper bounds on the numbers of Voronoi vertices and edges of a multiply connected polygon. The average numbers of Voronoi vertices and edges on the boundary of a Voronoi region will also be studied.

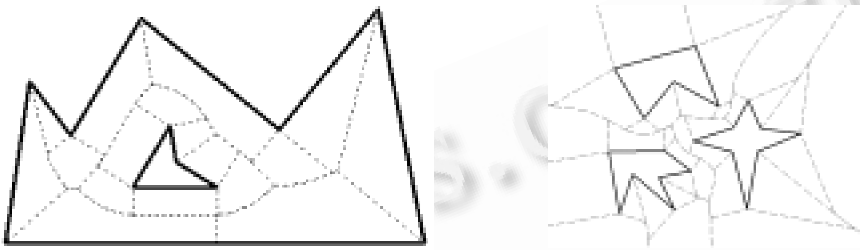
2 Basic Concepts

2.1 Polygon

First, some notation and definitions used in Ref.[1] are introduced. A *segment* is an open straight line segment.

A point or a segment is called an *object*. A *simply connected polygon* is a planar shape bounded by exactly one simple closed curve, called a *boundary*, consisting of segments. A *multiply connected polygon* is a planar shape bounded by several non-intersecting simple closed curves, called boundaries.

A multiply connected polygon has more than one boundary, and there is no intersection between the boundaries. One kind of multiply connected polygon has an outmost boundary called a *border contour*, and all other boundaries are inside the border contour and called the *island contours*. An island contour is also called a pocket in mechanical manufacturing. The second kind of multiply-connected polygons have no outmost boundary containing all other boundaries (see Fig.2); the interior of such a polygon is unbounded.



(a) Inner Voronoi diagram (b) Outer Voronoi diagram

Fig.2 Polygons and their Voronoi diagram

For the first kind of polygons, the Voronoi diagram partitions the interior of the border contours and the exterior of the island contours. The corresponding Voronoi diagram is called the *inner Voronoi diagram*. For the second kind of polygons, an *outer Voronoi diagram* is created. The outer VD partitions the whole plane except for the interior of every contour. The typical application of an inner Voronoi diagram is for modeling a pocket in NC path planning, and the outer Voronoi diagram is a useful representation in collision detection. Figure 2(a) shows the first kind of polygon in solid lines and its inner Voronoi diagram in dotted lines. Figure 2(b) shows the second kind of polygons in solid lines and its outer Voronoi diagram in dotted lines.

For pocket modeling it is assumed that a border contour is oriented *counter-clockwise* and island contours are oriented *clockwise*. Hence, the polygon lies on the left side of each contour for a traveler going along a contour.

Definition 1. A vertex v of a polygon is said to be reflex if its internal angle between the segments incident from v is greater than π; it is called a flat vertex if the internal angle is $=\pi$, and a convex vertex otherwise.

2.2 Voronoi diagram of polygon

We now introduce the notation of Voronoi diagrams of polygons. Let *Bisector* $b(o_1,o_2)$ denote the locus of all points equidistant from o_1 and o_2. Let $h(o_1,o_2)$ denote the set of points closer to o_1 than to o_2. Given a set O of objects in a planar domain, *the Voronoi region* $VR(o_1)$ of an object $o_1\in O$ is the set of all points closer to o_1 than to any other objects in O, i.e.

$$VR(o_1)=\bigcap_{o_2\in O-\{o_1\}} h(o_1,o_2)\ .$$

Given a polygon P in a planar domain, let O be the set of vertices and edges (i.e. segments) of P. The *Voronoi diagram of P* is given by

$$VD(O)=\bigcup_{o_1\in O} VR(o_1)\ .$$

In the Voronoi diagram of P, the common boundary of two adjacent regions is called a *Voronoi edge*. The points where Voronoi edges meet are called a *Voronoi vertex*. The *degree* of a Voronoi vertex v is the number of Voronoi edges incident at v.

3 Previous Work

For a multiply connected polygon P, Held[1] proves that the upper bounds of the numbers of vertices and edges of P's Voronoi diagram are $n+k-2$ and $2(n+k)-3$ respectively, where n is the number of P's vertices and k is the number of reflex vertices.

However, the proof in Ref.[1] (refer to the proof of Theorem 5.1[1]) does not consider the case of multiply connected polygon. The proof is based on the formula

$$2|E|\geq 3(|F|-1)+n+k$$

where $|E|$ is the number of edge of the Voronoi diagram, and $|F|$ is the number of faces of the planar graph constructed by the Voronoi diagram and the original polygon including the unbounded face.

For multi-connected polygon the formula should be modified to

$$2|E|\geq 3(|F|-h-1)+n+k$$

where h is the number of island contours. Held's formula therefore cannot be used for a multiply connected polygon.

4 Properties of Inner Voronoi Diagram

The following lemma is trivial.

Lemma 1. Let j be the number of non-leaf node, and i be the number of leaf nodes in a rooted tree. Suppose that every non-leaf node has at least 2 children, and the root node has at least 3 children. Then

$$j\leq i-2.$$

In particular, if every non-leaf node has exactly 2 children and the root node has exactly 3 children, then

$$j=i-2.$$

Definition 2. A **V-ring** C of an island contour B of a multiply connected polygon P is the smallest closed polygon form by the Voronoi edges of P such that C contains B but does not intersect B.

For example, in Fig.3, the V-ring of the island contour $p_1p_2p_3p_1$ consists of the Voronoi edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_{10}, $v_{10}v_{11}$, $v_{11}v_1$.

Definition 3. The **R-ring** of a reflex vertex v is the simple polygon consisting of the Voronoi edges of $VR(v)$.

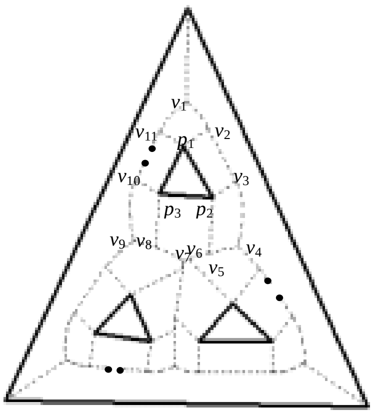


Fig.3 An inner Voronoi diagram and cutting points

For example, in Fig.3 (The polygon is shown in solid lines and its inner Voronoi diagram is shown in dotted lines. Solid dots stand for cutting points), the R-ring of the reflex vertex p_1 consists of the Voronoi edges v_1v_2, v_2p_1, p_1v_{11} and $v_{11}v_1$. Clearly, a reflex v is the shared endpoint of two Voronoi edges of $VR(v)$. Note that the R-ring is defined only for a reflex vertex.

Theorem 1. For the inner Voronoi diagram of a polygon P with h island contours, there is

$$m\leq n+k+2h-2.$$

Proof: The proof is trivial if $h=k=0$. So, suppose that $k>0$ or $h>0$. Then the Voronoi diagram of P is not a tree, since there are k R-rings or h V-rings. To make the Voronoi diagram a tree, we first modify the polygon by splitting every reflex vertex into two vertices and linking the new two vertices by a sufficiently short line segment.

Then we cut a Voronoi edge on every V-ring at the middle of the edge to produce cutting points (see Fig.3). In this way, the Voronoi diagram of *P* is turned into a tree**,** if *h* properly chosen edges are cut as follows: if a V-ring has one Voronoi edge that is not shared by other V-rings, then we can choose this Voronoi edge as the V-ring's cut edge. If every Voronoi edge of a V-ring *Vr* is shared by other V-rings, we first choose one Voronoi edge *e* of *Vr* and find the V-ring V_{r_1} that shares *e*. If V_{r_1} has one Voronoi edge e_1 that is not shared by other V-rings, we can cut e_1 and then cut *e*. Otherwise, we find the V-ring V_{r_2} that shares e_1. Let V_{r_1} be V_{r_2} and we repeatedly process V_{r_1} until we find a Voronoi edge of V_{r_1} that can be cut. The process must stop at one step for the inner Voronoi diagram of a polygon.

Take one Voronoi vertex as the root node and other Voronoi vertices as non-leaf nodes. All vertices of the modified polygon and the cutting points are leaf nodes, and all Voronoi edges are edges in the rooted tree. In this case the modified polygon has $n+k$ vertices, the rooted tree has $n+k+2h$ leaf nodes, *m* non-leaf nodes and $e+h$ edges.

Because the degree of every Voronoi vertex is at least 3[1], the root node of the rooted tree has at least 3 children, and the other non-leaf nodes have at least 2 children. By Lemma 1, we have

$$m \le n+k+2h-2.$$

This completes the proof.

Theorem 2. For the inner Voronoi diagram of a polygon *P* with *h* island contours, there is

$$e \le 2(n+k)+3h-3.$$

Proof: In the proof of Theorem 1, the rooted tree has $n+k+2h$ leaf nodes, *m* non-leaf nodes and $e+h$ edges. Since the number of nodes is one great than the number of edges, we have

$$e+h=(m+n+k+2h)-1,$$
$$e=m+n+k+h-1.$$

By Theorem 1,

$$e \le (n+k-2+2h)+n+k+h-1,$$
$$e \le 2(n+k)+3h-3.$$

Corollary 1. For a Voronoi region V_1 of the inner Voronoi diagram of a polygon *P*,

1) the average number a_e of the edges of V_1 is less than 5;

2) the average number a_v of the vertices of V_1 is less than 4.

Proof: Because a unique Voronoi region is defined for each edge or reflex vertex, the Voronoi diagram of *P* has $n+k$ Voronoi regions. Since two adjacent Voronoi regions share one Voronoi edge, we have

$$a_e \times (n+k)=2e.$$

By Theorem 2,

$$a_e \times (n+k) \le 2(2(n+k)+3h-3) \le 4(n+k)+6h-6,$$
$$a_e \le 4+6h/(n+k)-6/(n+k).$$

In a polygon with $h \ge 0$ island contours, for every island contour, the number of its vertices is greater than or equal to 3 and the number of its reflex vertices is also greater than or equal to 3. Therefore, $n+k>6h$. Then

$$a_e<4+(n+k)/(n+k)-6/(n+k)<5-6/(n+k).$$

It follows that $a_e<5$.

Since, for a Voronoi region, the number of Voronoi vertices is equal to the number of Voronoi edges minus one, we obtain $a_v=a_e-1<4$. The proof is completed.

If a polygon is a simply connected polygon, i.e. h=0, then we have the following theorem.

Theorem 3. For the inner Voronoi diagram of a simply connected polygon,

1) $m \le n+k-2$;

2) $e \le 2(n+k)-3$.

Corollary 2. For a Voronoi region of the inner Voronoi diagram of a simply connected polygon,

1) the average number a_e of Voronoi edges on its boundary is less than 4;

2) the average number a_v of Voronoi vertices on its boundary is less than 3.

Proof: By Theorem 3,

$$a_e \times (n+k) = 2e \le 2(2(n+k)-3) \le 4(n+k)-6,$$

$$a_e \le 4-6/(n+k).$$

Hence, $a_e<4$ and $a_v<3$.

5 Further Discussions

Let NL denote the set of all non-leaf nodes of a rooted tree, and q_n be the number of children of a non-leaf node n.

Lemma 2. Let j, i be the number of non-leaf nodes and the number of leaf nodes of a rooted tree. If every non-leaf node has at least 2 children, then

$$j = i - 1 - \sum_{n \in NL} (q_n - 2).$$

If the degree of every Voronoi vertex is considered, then, by Lemma 2, more precise equalities can be obtained as follows.

Theorem 4. For the inner Voronoi diagram of a polygon with $h \ge 0$ island contours inside, we have

1) $m = n + k - 1 + 2h - \sum_{n \in NL} (q_n - 2)$;

2) $e = 2(n + k) - 2 + 3h - \sum_{n \in NL} (q_n - 2)$.

Proof: By Lemma 2, these equalities can be proved by applying the same methods for proving Theorem 1 and Theorem 2.

6 Conclusions

The number of vertices and edges of the VD of a polygon is important in analyzing the complexity of VD-based algorithms. It's well known that the Voronoi diagram of a simply connected polygon has at most $n+k-2$ Voronoi vertices and at most $2(n+k)-3$ edges. But M. Held claims that these upper bounds also hold for a multiply-connected polygon[1].

We have shown that the upper bounds given by Held on the number of vertices and the number of edges of a Voronoi diagram do not hold for the inner Voronoi diagrams of multiply connected polygons, and we have proved new upper bounds for these cases—the Voronoi diagram of a polygon has at most $n+k+2h-2$ Voronoi vertices and at most $2(n+k)+3h-3$ edges. The average numbers of Voronoi vertices and edges on the boundary of a Voronoi region are also presented—for a Voronoi region of the inner Voronoi diagram of a polygon, the average number of Voronoi edges and Voronoi vertices are less than 5 and 4 respectively.

The result of this paper has been used to analyze the complexity of VD-based visibility computing algorithm in SDU Virtual Museum[12]. And inspired by the method of this paper we have also given the upper bounds of the size of outer Voronoi diagram of polygon[20].

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